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Symmetries of differential–difference dynamical systems in a two-dimensional lattice

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Abstract

The classification of differential–difference equations of the form $\ddot{u}_{nm} = F_{nm}(t, \{u_{pq}\}|_{(p,q)\in\Gamma})$ is considered according to their Lie point symmetry groups. The set Γ represents the point (n, m) and its six nearest neighbors in a two-dimensional triangular lattice. It is shown that the symmetry group can be at most 12 dimensional for Abelian symmetry algebras and 13 dimensional for nonsolvable symmetry algebras.

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1. Introduction

The purpose of this paper is to perform a symmetry analysis of the following class of differential–difference equations

$$\Delta_{nm} \equiv \ddot{u}_{nm} - F_{nm}(t, \{u_{pq}\}|_{(p,q)\in\Gamma}) = 0, \tag{1}$$

where the overdots denote the time derivative. The set Γ being the point (n, m) and its six nearest neighbors in a two-dimensional triangular lattice (see figure 1) given by

$$\Gamma = \{(n, m), (n + 1, m), (n, m + 1), (n - 1, m + 1), (n - 1, m), (n, m - 1), (n + 1, m - 1)\}.$$

The dependent variable $u_{nm}(t)$ can be interpreted as atomic normal displacement from their equilibrium position to the two-dimensional triangular lattice, where (n, m) locate the vertex on the skew coordinate system. The function F_{nm}, called the *interaction*, is an *a priori* unspecified smooth functions. Our aim is to classify such a system according to the Lie point symmetries that it allows, that is, to classify the functions F_{nm}.

The assumptions for this model are the following:

(i) The interaction F_{nm} involves only nearest neighbors in the triangular lattice, i.e. the atom (n, m) interacts only with the atoms of the set Γ, see figure 1.

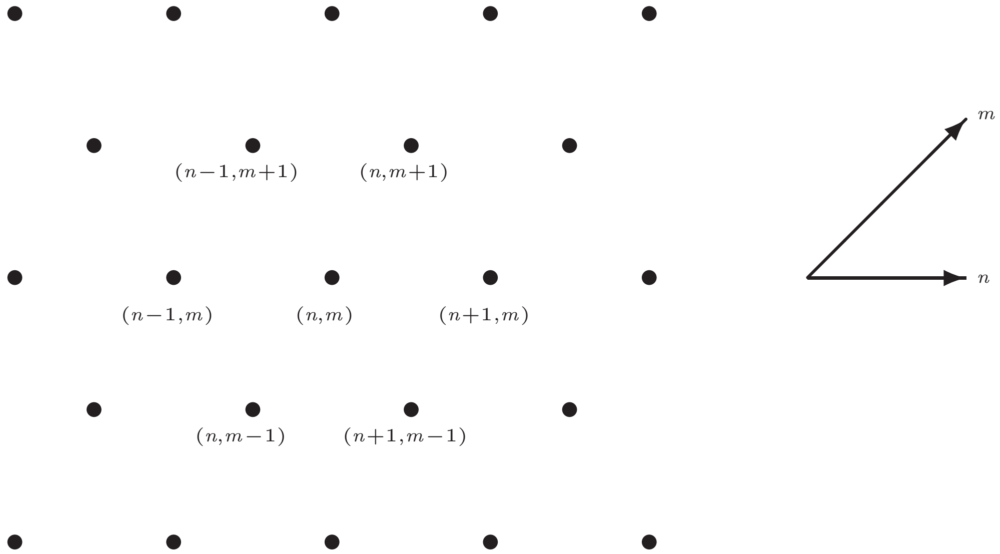

Figure 1. The atoms (n, m) and their six neighbors in a two-dimensional triangular lattice.

(ii) In the bulk of the paper the interaction F_{nm} is assumed to be nonlinear and coupled, i.e.

$$\frac{\partial^2 F_{nm}}{\partial u_{pq}\partial u_{p'q'}} \neq 0 \quad \text{for some } (p, q), (p', q') \in \Gamma$$

$$\text{and} \qquad \frac{\partial F_{nm}}{\partial u_{pq}} \neq 0 \quad \text{for all} \quad (p, q) \in \Gamma.$$

(iii) The interaction F_{nm} is isotropic in the three directions of the lattice.

(iv) We suppose that interaction F_{nm} depends continuously on the discrete variables n and m. For instance, interactions depending on terms of the type $(-1)^{nm}$ are excluded.

(v) Only 'maximal' symmetry algebras for a given interaction F_{nm} are listed. In other words, if a *given* interaction F_{nm} allows symmetry algebras $\mathcal{L}_1, \ldots, \mathcal{L}_N$ with $\dim \mathcal{L}_1 < \dim \mathcal{L}_2 < \cdots < \dim \mathcal{L}_N$, then we will only list the case $\mathcal{L}_N$.

Our motivation is the same as for classifying differential equations according to their symmetries, see [1–4]. When (1) allows a nontrivial symmetry group, then it is usually possible to obtain exact analytical solutions satisfying certain symmetry requirements. Models of this type have many applications in solid-state physics, see for instance the work of Büttner *et al* [5–8].

The formalism used in this paper was called the 'intrinsic method' [9, 10]. It has already been applied by Winternitz *et al* in the following one-dimensional cases: monoatomic molecular chains [11], diatomic molecular chains [12] and to a model with two types of atoms distributed along a double chain [13]. In this paper, we consider a two-dimensional case. In this formalism the Lie algebra of the symmetry group, often called 'symmetry algebra', is realized by the following vector fields

$$\widehat{X} = \tau(t, u_{nm})\partial_t + \phi_{nm}(t, u_{nm})\partial_{u_{nm}}. \tag{2}$$

The algorithm for finding the functions τ and ϕ_{nm} is to find the second prolongation $\mathrm{pr}^{(2)}\widehat{X}$ of $\widehat{X}$ and to impose that it should annihilate (1) on their solution set, i.e.

$$\mathrm{pr}^{(2)}\widehat{X} \cdot \Delta_{nm}|_{\Delta_{nm}=0} = 0. \tag{3}$$

Our prime objective will be to find and classify all interactions F_{nm} for which (1) allows at least a one-dimensional symmetry algebra. Thereafter, we will specify interactions further and find all those that allow a higher dimensional symmetry algebra.

Note that there are other formalisms to find the symmetry algebra of differential–difference equations. Consequently the use of the intrinsic method—applied for fixed lattice and used here—is not obligatory. Another method, called the 'differential equation approach to differential–difference equation' [10], can also be considered for such kind of lattices. Other methods exist in which the group transformations can also act on the lattice [14–17]. A complete review of these different methods has been considered in [18].

In section 2, we present the determining equations for the symmetries and illustrate the allowed transformation, i.e. the classification group that will be considered to obtain a class of equations. In section 3, all the calculations to find the one-dimensional symmetry algebras are explicitly given. Some notations are also introduced for the rest of the paper. Section 4 is devoted to interactions with Abelian symmetry algebras. They are denoted by A_{jk}, where the first index shows the dimension of the algebra and the second index enumerates the nonequivalent classes. Their dimensions satisfy $1 \leqslant \dim \mathcal{L} \leqslant 12$ with $\dim \mathcal{L} \neq 9, 11$. Nonsolvable symmetry algebras, denoted by NS_{jk}, are treated in section 5. They all contain $\mathrm{sl}(2,\mathbb{R})$ as subalgebra and $3 \leqslant \dim \mathcal{L} \leqslant 13$ with $\dim \mathcal{L} \neq 10, 12$. The results are summed up in section 6 where we also outline future work to be done.

2. Formulation of the problem

The second prolongation of the vector field (2) is given by [9–11]

$$\mathrm{pr}^{(2)} \widehat{X} = \tau(t, u_{nm})\partial_t + \sum_{(p,q)\in\Gamma} \phi_{pq}(t, u_{pq})\partial_{u_{pq}} + \phi^{tt}_{nm}\partial_{\ddot{u}_{nm}}. \tag{4}$$

The coefficient ϕ^{tt}_{nm} is a function depending on $n, m, t, u_{nm}, \dot{u}_{nm}$ and $\ddot{u}_{nm}$ given by

$$\phi^{tt}_{nm} = D_t^2\phi_{nm} - \left(D_t^2\tau\right)\dot{u}_{nm} - 2(D_t\tau)\ddot{u}_{nm}, \tag{5}$$

where D_t represents the total time derivative.

From (3)–(5) one can obtain the determining equations for the symmetries. We eliminate the $\ddot{u}_{nm}$ terms using (1) and then request that the coefficients of $\dot{u}^3_{nm}, \dot{u}^2_{nm}, \dot{u}_{nm}$ and $\dot{u}^0_{nm}$ should vanish independently. From the determining equations of the coefficients $\dot{u}^k_{nm} k = 1, 2, 3$, the vector field (2) must have the form

$$\widehat{X} = \tau(t)\partial_t + \left[\left(\frac{\dot{\tau}}{2} + a_{nm}\right)u_{nm} + \lambda_{nm}(t)\right]\partial_{u_{nm}}, \tag{6}$$

where $\dot{a}_{nm} = 0$. The remaining determining equation involves explicitly the interaction F_{nm} and is given by

$$\frac{\dddot{\tau}}{2}u_{nm} + \ddot{\lambda}_{nm} + \left(a_{nm} - \frac{3}{2}\dot{\tau}\right)F_{nm} - \tau\partial_t F_{nm} = \sum_{(p,q)\in\Gamma}\left[\left(\frac{\dot{\tau}}{2} + a_{pq}\right)u_{pq} + \lambda_{pq}(t)\right]\partial_{u_{pq}}F_{nm}. \tag{7}$$

Our aim is to solve (7) with respect to both forms of the nonlinear equation and the symmetry field. For every nonlinear interaction F_{nm} we wish to find the corresponding maximal symmetry group. Since for any symmetry group there will be a whole class of nonlinear differential–difference equation related to each other by point transformations, we will look for the simplest element of a given class. Hence, we shall classify (1) into equivalence classes under the action of a group of 'allowed transformations'. We restrict the allowed transformations to be fiber preserving, i.e. to have the form

$$\tilde{t} = \tilde{t}(t), \qquad u_{nm}(t) = \Omega_{nm}(t, \tilde{u}_{nm}(\tilde{t})), \qquad (\tilde{n}, \tilde{m}) = (n, m),$$

where Ω_{nm} and $\tilde{t}$ are some locally smooth and monotonous (invertible) functions. Substituting these transformations into (1) and requiring that the form of the equation be preserved, we find

$$u_{nm}(t) = \dot{\tilde{t}}^{-1/2} P_{nm} \tilde{u}_{nm}(\tilde{t}) + Q_{nm}(t),$$

where $P_{nm} \neq 0$, $\dot{P}_{nm} = 0$ and $\dot{\tilde{t}} \neq 0$. The transformed system is then given by

$$\ddot{\tilde{u}}_{nm}(\tilde{t}) = \tilde{F}_{nm}(\tilde{t}, \{\tilde{u}_{pq}\}|_{(p,q)\in\Gamma}),$$

where

$$\tilde{F}_{nm} = \dot{\tilde{t}}^{-3/2} P_{nm}^{-1}(F_{nm} - \ddot{Q}_{nm}(t)) + \tfrac{1}{2}\dot{\tilde{t}}^{-3}\big(\dddot{\tilde{t}} - \tfrac{3}{2}\dot{\tilde{t}}^{-1}\ddot{\tilde{t}}^{-2}\big)\tilde{u}_{nm}(\tilde{t}).$$

The vector field given by (6) is transformed into

$$\widehat{X} = \tau(t)\dot{\tilde{t}}\partial_{\tilde{t}} + \left\{\left(\frac{\dot{\tau}}{2} + a_{nm} + \frac{1}{2}\dot{\tilde{t}}^{-1}\ddot{\tilde{t}}\tau\right)\tilde{u}_{nm} + \dot{\tilde{t}}^{1/2} P_{nm}^{-1}\left[\left(\frac{\dot{\tau}}{2} + a_{nm}\right) Q_{nm}(t) + \lambda_{nm}(t) - \tau\dot{Q}_{nm}\right]\right\}\partial_{\tilde{u}_{nm}}.$$

In other words, under the allowed transformations the vector field (6) characterized by the triplet $\{\tau(t), a_{nm}, \lambda_{nm}(t)\}$ is transformed into the triplet $\{\tilde{\tau}(\tilde{t}), \tilde{a}_{nm}, \tilde{\lambda}_{nm}(\tilde{t})\}$ where

$$\tau(t) \to \tilde{\tau}(\tilde{t}) = \tau(t(\tilde{t}))\dot{\tilde{t}},$$

$$a_{nm} \to \tilde{a}_{nm} = a_{nm},$$

$$\lambda_{nm}(t) \to \tilde{\lambda}_{nm}(\tilde{t}) = \dot{\tilde{t}}^{1/2} P_{nm}^{-1}\left[\left(\frac{\dot{\tau}}{2} + a_{nm}\right) Q_{nm}(t) + \lambda_{nm}(t) - \tau\dot{Q}_{nm}\right].$$

3. One-dimensional symmetry algebras of the systems

Let us now introduce some notations that will be useful in what follows. First, we will write functions g of the type $g(\{\xi_{pq}\}|_{(p,q)\in S})$, where S is a subset of $\mathbb{Z}^2$ and $\xi_{pq} : \mathbb{Z}^2 \to \mathbb{R}$, more simply as $g(\xi_{pq})$ with $(p, q) \in S$. Second, if $\{f_{pq}^{(1)}, f_{pq}^{(2)}, \ldots, f_{pq}^{(N)}\}$ is a set of N functions $f_{pq}^{(i)} : \mathbb{Z}^2 \to \mathbb{R}$ depending on the discrete variables p, q, then we define the determinant function $\mathcal{D} : \mathbb{Z}^{2N} \to \mathbb{R}$ by

$$\mathcal{D}\big[f_{p_1q_1}^{(1)}, f_{p_2q_2}^{(2)}, \ldots, f_{p_Nq_N}^{(N)}\big] := \begin{vmatrix} f_{p_1q_1}^{(1)} & f_{p_2q_2}^{(1)} & \cdots & f_{p_Nq_N}^{(1)} \\ f_{p_1q_1}^{(2)} & f_{p_2q_2}^{(2)} & \cdots & f_{p_Nq_N}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{p_1q_1}^{(N)} & f_{p_2q_2}^{(N)} & \cdots & f_{p_Nq_N}^{(N)} \end{vmatrix}.$$

For example, we have

$$\mathcal{D}[f_{nm}, g_{n+1m}, 1_{nm+1}] = \begin{vmatrix} f_{nm} & f_{n+1m} & f_{nm+1} \\ g_{nm} & g_{n+1m} & g_{nm+1} \\ 1_{nm} & 1_{n+1m} & 1_{nm+1} \end{vmatrix}$$

where the function $1_{pq} : \mathbb{Z}^2 \to \mathbb{R}$ is the constant function $(p, q) \mapsto 1$. (Here, the indices of the constant function are written only for clarity of the determinant function $\mathcal{D}[f_{nm}, g_{n+1m}, 1_{nm+1}]$.)

Theorem 3.1. *Equation (1) allows a one-dimensional symmetry algebra for three classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$$A_{1,1}: \quad \widehat{X} = \partial_t + a_{nm} u_{nm} \partial_{u_{nm}}, \tag{8}$$

$$F_{nm} = \exp(a_{nm} t) f_{nm}(\xi_{pq}), \qquad \xi_{pq} = u_{pq} \exp(-a_{pq} t), \qquad (p, q) \in \Gamma. \tag{9}$$

$$A_{1,2}: \quad \widehat{X} = a_{nm} u_{nm} \partial_{u_{nm}}, \tag{10}$$

$$F_{nm} = u_{nm} f_{nm}(t, \xi_{pq}), \qquad \xi_{pq} = \frac{(u_{nm})^{a_{pq}}}{(u_{pq})^{a_{nm}}}, \qquad (p, q) \in \Gamma \backslash \{(n, m)\}. \tag{11}$$

$$A_{1,3}: \quad \widehat{X} = \lambda_{nm}(t) \partial_{u_{nm}}, \tag{12}$$

$$F_{nm} = \frac{\ddot{\lambda}_{nm}}{\lambda_{nm}} u_{nm} + f_{nm}(t, \xi_{pq}), \qquad \xi_{pq} = \mathcal{D}[u_{nm}, \lambda_{pq}], \qquad (p, q) \in \Gamma \backslash \{(n, m)\}. \tag{13}$$

Proof. We suppose that the system (1) has at least a one-dimensional symmetry group generated by vector field of the form (6). Using the allowed transformations, we transform $\widehat{X}$ into three inequivalent classes:

(a) For the case $\tau(t) \neq 0$, we choose the functions $\tilde{t}(t)$ and $Q_{nm}(t)$ so as to transform $\tau(t) \to 1$ and $\lambda_{nm}(t) \to 0$. More precisely, we look for functions $\tilde{t}(t)$ and $Q_{nm}(t)$ satisfying the following ODEs:

$$\tau(t)\dot{\tilde{t}} = 1, \qquad \tau \dot{Q}_{nm} - \lambda_{nm}(t) - \left(\frac{\dot{\tau}}{2} + a_{nm}\right) Q_{nm}(t) = 0.$$

Hence, under allowed transformations, the vector field (6) with $\tau(t) \neq 0$ is given by (8). We can now solve the remaining equation (7), for $\tau = 1$ and $\lambda_{nm} = 0$, by applying the method of characteristics and we find function (9).

(b) For the case $\tau = 0$ and $a_{nm} \neq 0$ we choose the function $Q_{nm}(t)$ such that $\lambda_{nm}(t) \to 0$, i.e.

$$a_{nm} Q_{nm}(t) + \lambda_{nm}(t) = 0.$$

The vector field (6) is then given as (10). The remaining equation (7) gives (11).

(c) Finally, when $\tau = 0$ and $a_{nm} = 0$ we already find the vector field (12) and the remaining equation (7) gives us (13). □

Let us note that vector field (12) can be simplified in some cases. Indeed, the vector field can be transformed as $\lambda_{nm}(t) \to \tilde{\lambda}_{nm}(\tilde{t}) = \dot{\tilde{t}}^{1/2} P_{nm}^{-1} \lambda_{nm}(t)$ from the allowed transformations (8). Hence, if $\lambda_{nm}(t)$ is separable in terms of the discrete variables n, m and the continuous variable t, then we can transform $\lambda_{nm}(t)$ into 1.

We observe that the existence of a one-dimensional symmetry algebra restricts the interaction F_{nm} to arbitrary functions of seven variables, rather than eight variables in the original equation (1). In the next sections we will assume that the interaction F_{nm} and one of the symmetry generators is already in 'canonical form', i.e. they have the form (8), (10) or (12) with the corresponding interaction. We will illustrate how the interaction is further restricted by the existence of the higher dimensional symmetry algebra.

4. Abelian symmetry algebras

Theorem 4.1. *Equation (1) allows a two-dimensional Abelian symmetry algebra for four classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{2,1}$: $\widehat{X}_1 = \partial_t + a^{(1)}_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_2 = a^{(2)}_{nm} u_{nm}\partial_{u_{nm}}$,

$$F_{nm} = u_{nm} f_{nm}(\xi_{pq}), \quad \xi_{pq} = \frac{(u_{pq})^{a^{(2)}_{nm}}}{(u_{nm})^{a^{(2)}_{pq}}} \exp\left(\mathcal{D}\left[a^{(1)}_{nm}, a^{(2)}_{pq}\right] t\right), \quad (p,q) \in \Gamma\backslash\{(n,m)\}.$$

$A_{2,2}$: $\widehat{X}_1 = \partial_t + a_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_2 = \mathrm{e}^{a_{nm}t}\partial_{u_{nm}}$,

$$F_{nm} = a^2_{nm} u_{nm} + \mathrm{e}^{a_{nm}t} f_{nm}(\xi_{pq}), \quad \xi_{pq} = \mathcal{D}[u_{nm}\,\mathrm{e}^{-a_{nm}t}, 1_{pq}], \quad (p,q) \in \Gamma\backslash\{(n,m)\}.$$

$A_{2,3}$: $\widehat{X}_1 = a^{(1)}_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_2 = a^{(2)}_{nm} u_{nm}\partial_{u_{nm}}$,

$$F_{nm} = u_{nm} f_{nm}(t, \xi_{pq}), \qquad (p,q) \in \Gamma\backslash\{(n,m), (n+1,m)\},$$

$$\xi_{pq} = (u_{nm})^{-\mathcal{D}[a^{(1)}_{n+1m}, a^{(2)}_{pq}]} (u_{n+1m})^{\mathcal{D}[a^{(1)}_{nm}, a^{(2)}_{pq}]} (u_{pq})^{-\mathcal{D}[a^{(1)}_{nm}, a^{(2)}_{n+1m}]}.$$

$A_{2,4}$: $\widehat{X}_1 = \partial_{u_{nm}}$, $\widehat{X}_2 = t\partial_{u_{nm}}$,

$$F_{nm} = f_{nm}(t, \xi_{pq}), \qquad \xi_{pq} = \mathcal{D}[u_{nm}, 1_{pq}], \qquad (p,q) \in \Gamma\backslash\{(n,m)\}.$$

Two cases are not listed in these two-dimensional symmetry algebras. One case corresponds to interaction F_{nm} with $\widehat{X}_1$ and $\widehat{X}_2$ of $A_{4,4}$ (see above in the list of four-dimensional symmetry algebras) and is not listed here since the symmetry algebra can be of dimension 4 with the same interaction, i.e. the algebra is not 'maximal'. The second case corresponds to the degenerate case $\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}\right] = 0$ with $\widehat{X}_1$ and $\widehat{X}_2$ of $A_{4,4}$. The generators are then given by $\widehat{X}_1 = \lambda_{nm}(t)\partial_{u_{nm}}$ and $\widehat{X}_2 = \gamma_m(t)\lambda_{nm}(t)\partial_{u_{nm}}$ with $\gamma_m \neq \gamma_{m+1}$ and $\dot{\gamma}_m$. Hence, we obtain a non-isotropic system.

Theorem 4.2. *Equation (1) allows a three-dimensional Abelian symmetry algebra for three classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{3,1}$: $\widehat{X}_1 = \partial_t + a^{(1)}_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_2 = a^{(2)}_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_3 = a^{(3)}_{nm} u_{nm}\partial_{u_{nm}}$,

$$F_{nm} = u_{nm} f_{nm}(\xi_{pq}), \qquad (p,q) \in \Gamma\backslash\{(n,m), (n+1,m)\},$$

$$\xi_{pq} = (u_{nm})^{\mathcal{D}[a^{(2)}_{n+1m}, a^{(3)}_{pq}]} (u_{n+1m})^{-\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{pq}]} (u_{pq})^{\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{n+1m}]} \exp\left(-\mathcal{D}\left[a^{(1)}_{nm}, a^{(2)}_{n+1m}, a^{(3)}_{pq}\right] t\right).$$

$A_{3,2}$: $\widehat{X}_1 = \partial_t + a_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_2 = \mathrm{e}^{a_{nm}t}\partial_{u_{nm}}$, $\widehat{X}_3 = \kappa_{nm}\,\mathrm{e}^{a_{nm}t}\partial_{u_{nm}}$,

$$F_{nm} = a^2_{nm} u_{nm} + \mathrm{e}^{a_{nm}t} f_{nm}(\xi_{pq}), \qquad (p,q) \in \Gamma\backslash\{(n,m), (n+1,m)\}$$

$$\xi_{pq} = \mathcal{D}[u_{nm}\,\mathrm{e}^{-a_{nm}t}, \kappa_{n+1m}, 1_{pq}], \qquad \dot{\kappa}_{nm} = 0, \qquad \kappa_{nm} \neq \kappa_{n+1m}, \qquad \kappa_{nm} \neq \kappa_{nm+1}.$$

$A_{3,3}$: $\widehat{X}_1 = a^{(1)}_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_2 = a^{(2)}_{nm} u_{nm}\partial_{u_{nm}}$, $\widehat{X}_3 = a^{(3)}_{nm} u_{nm}\partial_{u_{nm}}$,

$$F_{nm} = u_{nm} f_{nm}(t, \xi_{pq}), \qquad (p,q) \in \Gamma\backslash\{(n,m), (n+1,m), (n,m+1)\},$$

$$\xi_{pq} = (u_{nm})^{\mathcal{D}[a^{(1)}_{n+1m}, a^{(2)}_{nm+1}, a^{(3)}_{pq}]} (u_{n+1m})^{-\mathcal{D}[a^{(1)}_{nm}, a^{(2)}_{nm+1}, a^{(3)}_{pq}]} (u_{nm+1})^{\mathcal{D}[a^{(1)}_{nm}, a^{(2)}_{n+1m}, a^{(3)}_{pq}]} \times (u_{pq})^{-\mathcal{D}[a^{(1)}_{nm}, a^{(2)}_{n+1m}, a^{(3)}_{nm+1}]}.$$

Again here, four three-dimensional Lie algebras are not listed below. Three of them are not listed because they are not 'maximal'. The corresponding maximal algebras being given by $A_{6,4}$, $A_{4,4}$ and $A_{4,5}$ above. The other algebra corresponds to the degenerate case $\lambda_{n+1m} = \lambda_{nm}$ of $A_{4,5}$ (without $\widehat{X}_4$). The generators are then given by $\widehat{X}_1 = \partial_{u_{nm}}$, $\widehat{X}_2 = t\partial_{u_{nm}}$ and $\widehat{X}_3 = \lambda_m(t)\partial_{u_{nm}}$ with $\lambda_{m+1} \neq \lambda_m$. Hence, we obtain a non-isotropic system.

The discrete function κ_{nm} in $A_{3,2}$ depends on n and m; otherwise we obtain a decoupled non-isotropic system.

Theorem 4.3. *Equation (1) allows a four-dimensional Abelian symmetry algebra for five classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{4,1}$: $\widehat{X}_1 = \partial_t + a_{nm}^{(1)} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_i = a_{nm}^{(i)} u_{nm} \partial_{u_{nm}}, \quad i = 2, 3, 4$

$$F_{nm} = u_{nm} f_{nm}(\xi_{pq}), \qquad (p, q) \in \Gamma \backslash \{(n, m), (n+1, m), (n, m+1)\},$$

$$\xi_{pq} = (u_{nm})^{\mathcal{D}[a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{pq}^{(4)}]} (u_{n+1m})^{-\mathcal{D}[a_{nm}^{(2)}, a_{nm+1}^{(3)}, a_{pq}^{(4)}]} (u_{nm+1})^{\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{pq}^{(4)}]}$$
$$\times (u_{pq})^{-\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{nm+1}^{(4)}]} \exp\left(-\mathcal{D}\left[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{pq}^{(4)}\right] t\right).$$

$A_{4,2}$: $\widehat{X}_1 = \partial_t + a_{nm} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_2 = \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \kappa_{nm}^{(i)} \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \qquad i = 1, 2$

$$\dot{\kappa}_{nm}^{(i)} = 0, \qquad \kappa_{n+1m}^{(i)} \neq \kappa_{nm}^{(i)}, \qquad \kappa_{nm+1}^{(i)} \neq \kappa_{nm}^{(i)},$$

$$F_{nm} = a_{nm}^2 u_{nm} + \mathrm{e}^{a_{nm} t} f_{nm}(\xi_{pq}), \qquad (p, q) \in \Gamma \backslash \{(n, m), (n+1, m), (n, m+1)\},$$

$$\xi_{pq} = \mathcal{D}\left[u_{nm}\, \mathrm{e}^{-a_{nm} t}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, 1_{pq}\right].$$

$A_{4,3}$: $\widehat{X}_i = a_{nm}^{(i)} u_{nm} \partial_{u_{nm}}, \qquad i = 1, \ldots, 4$

$$F_{nm} = u_{nm} f_{nm}(t, \xi_{pq}), \qquad (p, q) \in \{(n-1, m), (n, m-1), (n+1, m-1)\},$$

$$\xi_{pq} = (u_{nm})^{-\mathcal{D}[a_{n+1m}^{(1)}, a_{nm+1}^{(2)}, a_{n-1m+1}^{(3)}, a_{pq}^{(4)}]} (u_{n+1m})^{\mathcal{D}[a_{nm}^{(1)}, a_{nm+1}^{(2)}, a_{n-1m+1}^{(3)}, a_{pq}^{(4)}]}$$
$$\times (u_{nm+1})^{-\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{n-1m+1}^{(3)}, a_{pq}^{(4)}]} (u_{n-1m+1})^{\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{pq}^{(4)}]}$$
$$\times (u_{pq})^{-\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}]}.$$

$A_{4,4}$: $\widehat{X}_i = \lambda_{nm}^{(i)}(t) \partial_{u_{nm}}, \qquad i = 1, 2; \qquad \widehat{Y}_k = \left(\sum_{j=1}^{2} \omega_{kj}(t) \lambda_{nm}^{(j)}(t)\right) \partial_{u_{nm}}, \qquad k = 1, 2$

$$F_{nm} = 1 + \begin{vmatrix} u_{nm} & 1 & u_{n+1m} \\ \lambda_{nm}^{(1)} & \ddot{\lambda}_{nm}^{(1)} & \lambda_{n+1m}^{(1)} \\ \lambda_{nm}^{(2)} & \ddot{\lambda}_{nm}^{(2)} & \lambda_{n+1m}^{(2)} \end{vmatrix} \left(\mathcal{D}\left[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}\right]\right)^{-1} + f_{nm}(t, \xi_{pq}),$$

$$\xi_{pq} = \mathcal{D}\left[u_{nm}, \lambda_{n+1m}^{(1)}, \lambda_{pq}^{(2)}\right], \quad (p, q) \in \Gamma \backslash \{(n, m), (n+1, m)\}, \quad \mathcal{D}\left[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}\right] \neq 0.$$

$$\text{with} \quad \sum_{j=1}^{2} \left(\ddot{\omega}_{kj} \lambda_{nm}^{(j)} + 2\dot{\omega}_{kj} \dot{\lambda}_{nm}^{(j)}\right) = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0.$$

$A_{4,5}$: $\widehat{X}_1 = \partial_{u_{nm}}, \qquad \widehat{X}_2 = t \partial_{u_{nm}}, \qquad \widehat{X}_3 = \lambda_{nm}(t) \partial_{u_{nm}}, \qquad \widehat{X}_4 = (\omega(t) \lambda_{nm}(t) + \sigma(t)) \partial_{u_{nm}},$

$$F_{nm} = \frac{\ddot{\lambda}_{nm}}{\lambda_{n+1m} - \lambda_{nm}} (u_{n+1m} - u_{nm}) + f_{nm}(t, \xi_{pq}), \qquad \xi_{pq} = \mathcal{D}[u_{nm}, \lambda_{n+1m}, 1_{pq}],$$

$$\dot{\omega} > 0, \qquad \lambda_{nm}(t) = \frac{1}{\sqrt{\dot{\omega}}} \left(c_{nm} - \frac{1}{2} \int_0^t \frac{\ddot{\sigma}(s)}{\sqrt{\dot{\omega}(s)}} \, \mathrm{d}s\right), \qquad \dot{c}_{nm} = 0,$$

$$\lambda_{n+1m} \neq \lambda_{nm}, \qquad (p, q) \in \Gamma \backslash \{(n, m), (n+1, m)\}.$$

Theorem 4.4. *Equation (1) allows a five-dimensional Abelian symmetry algebra for three classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{5,1}$:
$$\widehat{X}_1 = \partial_t + a_{nm}^{(1)} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_i = a_{nm}^{(i)} u_{nm} \partial_{u_{nm}}, \qquad i = 2, \ldots, 5$$
$$F_{nm} = u_{nm} f_{nm}(\xi_{pq}), \qquad (p, q) \in \{(n-1, m), (n, m-1), (n+1, m-1)\},$$
$$\xi_{pq} = (u_{nm})^{\mathcal{D}[a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{pq}^{(5)}]} (u_{n+1m})^{-\mathcal{D}[a_{nm}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{pq}^{(5)}]}$$
$$\times (u_{nm+1})^{\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{n-1m+1}^{(4)}, a_{pq}^{(5)}]} (u_{n-1m+1})^{-\mathcal{D}[a_{mn}^{(2)}, a_{n+1m}^{(3)}, a_{nm+1}^{(4)}, a_{pq}^{(5)}]}$$
$$\times (u_{pq})^{\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{nm+1}^{(4)}, a_{n-1m+1}^{(5)}]} \exp\left(-\mathcal{D}\left[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{pq}^{(5)}\right] t\right).$$

$A_{5,2}$:
$$\widehat{X}_1 = \partial_t + a_{nm} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_2 = \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \kappa_{nm}^{(i)} \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \quad i = 1, 2, 3$$
$$\dot{\kappa}_{nm}^{(i)} = 0, \qquad \kappa_{n+1m}^{(i)} \neq \kappa_{nm}^{(i)}, \qquad \kappa_{nm+1}^{(i)} \neq \kappa_{nm}^{(i)},$$
$$F_{nm} = a_{nm}^2 u_{nm} + \mathrm{e}^{a_{nm} t} f_{nm}(\xi_{pq}), \qquad (p, q) \in \{(n-1, m), (n, m-1), (n+1, m-1)\},$$
$$\xi_{pq} = \mathcal{D}\left[u_{nm} \mathrm{e}^{-a_{nm} t}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, 1_{pq}\right].$$

$A_{5,3}$:
$$\widehat{X}_i = a_{nm}^{(i)} u_{nm} \partial_{u_{nm}}, \qquad i = 1, \ldots, 5$$
$$F_{nm} = u_{nm} f_{nm}(t, \xi_{pq}), \qquad (p, q) \in \{(n, m-1), (n+1, m-1)\},$$
$$\xi_{pq} = (u_{nm})^{-\mathcal{D}[a_{n+1m}^{(1)}, a_{nm+1}^{(2)}, a_{n-1m+1}^{(3)}, a_{n-1m}^{(4)}, a_{pq}^{(5)}]} (u_{n+1m})^{\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{n-1m+1}^{(3)}, a_{n-1m}^{(4)}, a_{pq}^{(5)}]}$$
$$\times (u_{nm+1})^{-\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{n-1m+1}^{(3)}, a_{n-1m}^{(4)}, a_{pq}^{(5)}]} (u_{n-1m+1})^{\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m}^{(4)}, a_{pq}^{(5)}]}$$
$$\times (u_{n-1m})^{-\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{pq}^{(5)}]} (u_{pq})^{\mathcal{D}[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{n-1m}^{(5)}]}.$$

Theorem 4.5. *Equation (1) allows a six-dimensional Abelian symmetry algebra for five classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{6,1}$:
$$\widehat{X}_1 = \partial_t + a_{nm}^{(1)} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_i = a_{nm}^{(i)} u_{nm} \partial_{u_{nm}}, \qquad i = 2, \ldots, 6$$
$$F_{nm} = u_{nm} f_{nm}(\xi_{pq}), \qquad (p, q) \in \{(n, m-1), (n+1, m-1)\}$$
$$\xi_{pq} = (u_{nm})^{\mathcal{D}[a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{n-1m}^{(5)}, a_{pq}^{(6)}]} (u_{n+1m})^{-\mathcal{D}[a_{nm}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{n-1m}^{(5)}, a_{pq}^{(6)}]}$$
$$\times (u_{nm+1})^{\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{n-1m+1}^{(4)}, a_{n-1m}^{(5)}, a_{pq}^{(6)}]} (u_{n-1m+1})^{-\mathcal{D}[a_{mn}^{(2)}, a_{n+1m}^{(3)}, a_{nm+1}^{(4)}, a_{n-1m}^{(5)}, a_{pq}^{(6)}]}$$
$$\times (u_{n-1m})^{\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{nm+1}^{(4)}, a_{n-1m+1}^{(5)}, a_{pq}^{(6)}]} (u_{pq})^{-\mathcal{D}[a_{nm}^{(2)}, a_{n+1m}^{(3)}, a_{nm+1}^{(4)}, a_{n-1m+1}^{(5)}, a_{n-1m}^{(6)}]}$$
$$\times \exp\left(-\mathcal{D}\left[a_{nm}^{(1)}, a_{n+1m}^{(2)}, a_{nm+1}^{(3)}, a_{n-1m+1}^{(4)}, a_{n-1m}^{(5)}, a_{pq}^{(6)}\right] t\right).$$

$A_{6,2}$:
$$\widehat{X}_1 = \partial_t + a_{nm} u_{nm} \partial_{u_{nm}}, \quad \widehat{X}_2 = \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \quad \widehat{X}_{i+2} = \kappa_{nm}^{(i)} \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \quad i = 1, \ldots, 4$$
$$\dot{\kappa}_{nm}^{(i)} = 0, \qquad \kappa_{n+1m}^{(i)} \neq \kappa_{nm}^{(i)}, \qquad \kappa_{nm+1}^{(i)} \neq \kappa_{nm}^{(i)},$$
$$F_{nm} = a_{nm}^2 u_{nm} + \mathrm{e}^{a_{nm} t} f_{nm}(\xi_{pq}), \qquad (p, q) \in \{(n, m-1), (n+1, m-1)\},$$
$$\xi_{pq} = \mathcal{D}\left[u_{nm} \mathrm{e}^{-a_{nm} t}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, \kappa_{n-1m}^{(4)}, 1_{pq}\right].$$

$A_{6,3}$:
$$\widehat{X}_i = a_{nm}^{(i)} u_{nm} \partial_{u_{nm}}, \quad i = 1, \ldots, 6; \quad F_{nm} = u_{nm} f_{nm}(t, \xi),$$
$$\xi = (u_{nm})^{-\mathcal{D}[a_{n+1m}^{(1)}, a_{nm+1}^{(2)}, a_{n-1m+1}^{(3)}, a_{n-1m}^{(4)}, a_{nm-1}^{(5)}, a_{n+1,m-1}^{(6)}]} (u_{n+1m})^{\mathcal{D}[a_{nm}^{(1)}, a_{nm+1}^{(2)}, a_{n-1m+1}^{(3)}, a_{n-1m}^{(4)}, a_{nm-1}^{(5)}, a_{n+1m-1}^{(6)}]}$$

$$\times\ (u_{nm+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1},a^{(6)}_{n+1m-1}]}$$
$$\times\ (u_{n-1m+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1},a^{(6)}_{n+1m-1}]}$$
$$\times\ (u_{n-1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{nm-1},a^{(6)}_{n+1m-1}]}$$
$$\times\ (u_{nm-1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{n-1m},a^{(6)}_{n+1m-1}]}$$
$$\times\ (u_{n+1m-1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{n-1m},a^{(6)}_{nm-1}]}.$$

$A_{6,4}$: $\widehat{X}_i = \lambda^{(i)}_{nm}(t)\partial_{u_{nm}}, \qquad i = 1, 2, 3; \qquad \widehat{Y}_k = \left(\sum_{j=1}^{3} \omega_{kj}(t)\lambda^{(j)}_{nm}(t)\right)\partial_{u_{nm}}, \qquad k = 1, 2, 3$

$$F_{nm} = 1 + \begin{vmatrix} u_{nm} & 1 & u_{n+1m} & u_{nm+1} \\ \lambda^{(1)}_{nm} & \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{n+1m} & \lambda^{(1)}_{nm+1} \\ \lambda^{(2)}_{nm} & \ddot{\lambda}^{(2)}_{nm} & \lambda^{(2)}_{n+1m} & \lambda^{(2)}_{nm+1} \\ \lambda^{(3)}_{nm} & \ddot{\lambda}^{(3)}_{nm} & \lambda^{(3)}_{n+1m} & \lambda^{(3)}_{nm+1} \end{vmatrix} \left(\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}\right]\right)^{-1} + f_{nm}(t, \xi_{pq})$$

$$\xi_{pq} = \mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, \lambda^{(3)}_{pq}\right], \qquad (p, q) \in \Gamma\backslash\{(n, m), (n + 1, m), (n, m + 1)\},$$

$$\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}\right] \neq 0, \qquad \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}\right] \neq 0,$$

$$\text{with} \quad \sum_{j=1}^{3} \left(\ddot{\omega}_{kj}\lambda^{(j)}_{nm} + 2\dot{\omega}_{kj}\dot{\lambda}^{(j)}_{nm}\right) = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0.$$

$A_{6,5}$: $\widehat{X}_1 = \partial_{u_{nm}}, \qquad \widehat{X}_2 = t\,\partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \lambda^{(i)}_{nm}(t)\partial_{u_{nm}}, \qquad i = 1, 2$

$$\widehat{Y}_k = \left(\sum_{j=1}^{2} \omega_{kj}(t)\lambda^{(j)}_{nm}(t) + \sigma^{(k)}(t)\right)\partial_{u_{nm}}, \qquad k = 1, 2$$

$$F_{nm} = \frac{\ddot{\lambda}^{(1)}_{nm}}{\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}}(u_{n+1m} - u_{nm}) - \frac{\mathcal{D}_1 \cdot \mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1m}, 1_{nm+1}\right]}{\left(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\right) \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right]}$$
$$+ f_{nm}(t, \xi_{pq}),$$

$$\mathcal{D}_1 = \begin{vmatrix} \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{nm} & \lambda^{(1)}_{n+1m} \\ \ddot{\lambda}^{(2)}_{nm} & \lambda^{(2)}_{nm} & \lambda^{(2)}_{n+1m} \\ 0 & 1 & 1 \end{vmatrix}, \qquad \xi_{pq} = \mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1,m}, \lambda^{(2)}_{nm+1}, 1_{pq}\right],$$

$$(p, q) \in \Gamma\backslash\{(n, m), (n + 1, m), (n, m + 1)\},$$

$$\lambda^{(1)}_{n+1m} \neq \lambda^{(1)}_{nm}, \qquad \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right] \neq 0,$$

$$\text{with} \quad \sum_{j=1}^{2} \left(\ddot{\omega}_{kj}\lambda^{(j)}_{nm} + 2\dot{\omega}_{kj}\dot{\lambda}^{(j)}_{nm}\right) + \ddot{\sigma}^{(k)} = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0.$$

Theorem 4.6. *Equation (1) allows a seven-dimensional Abelian symmetry algebra for two classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{7,1}$: $\widehat{X}_1 = \partial_t + a^{(1)}_{nm} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_i = a^{(i)}_{nm} u_{nm} \partial_{u_{nm}}, \quad i = 2, \ldots, 7$

$$F_{nm} = u_{nm} f_{nm}(\xi),$$

$$\xi = (u_{nm})^{-\mathcal{D}[a^{(2)}_{n+1m}, a^{(3)}_{nm+1}, a^{(4)}_{n-1m+1}, a^{(5)}_{n-1m}, a^{(6)}_{nm-1}, a^{(7)}_{n+1,m-1}]} (u_{n+1m})^{\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{nm+1}, a^{(4)}_{n-1m+1}, a^{(5)}_{n-1m}, a^{(6)}_{nm-1}, a^{(7)}_{n+1,m-1}]}$$
$$\times (u_{nm+1})^{-\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{n+1m}, a^{(4)}_{n-1m+1}, a^{(5)}_{n-1m}, a^{(6)}_{nm-1}, a^{(7)}_{n+1,m-1}]}$$
$$\times (u_{n-1m+1})^{\mathcal{D}[a^{(2)}_{mn}, a^{(3)}_{n+1m}, a^{(4)}_{nm+1}, a^{(5)}_{n-1m}, a^{(6)}_{nm-1}, a^{(7)}_{n+1,m-1}]}$$
$$\times (u_{n-1m})^{-\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{n+1m}, a^{(4)}_{nm+1}, a^{(5)}_{n-1m+1}, a^{(6)}_{nm-1}, a^{(7)}_{n+1,m-1}]}$$
$$\times (u_{nm-1})^{\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{n+1m}, a^{(4)}_{nm+1}, a^{(5)}_{n-1m+1}, a^{(6)}_{n-1m} a^{(7)}_{n+1,m-1}]}$$
$$\times (u_{n+1,m-1})^{-\mathcal{D}[a^{(2)}_{nm}, a^{(3)}_{n+1m}, a^{(4)}_{nm+1}, a^{(5)}_{n-1m+1}, a^{(6)}_{n-1m}, a^{(7)}_{nm-1}]}$$
$$\times \exp\left(-\mathcal{D}\left[a^{(1)}_{nm}, a^{(2)}_{n+1m}, a^{(3)}_{nm+1}, a^{(4)}_{n-1m+1}, a^{(5)}_{n-1m}, a^{(6)}_{nm-1}, a^{(7)}_{n+1,m-1}\right]t\right).$$

$A_{7,2}$: $\widehat{X}_1 = \partial_t + a_{nm} u_{nm} \partial_{u_{nm}}, \qquad \widehat{X}_2 = \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \kappa^{(i)}_{nm} \mathrm{e}^{a_{nm} t} \partial_{u_{nm}}, \qquad i = 1, \ldots, 5$

$$F_{nm} = a^2_{nm} u_{nm} + \mathrm{e}^{a_{nm} t} f_{nm}(\xi), \qquad \dot{\kappa}^{(i)}_{nm} = 0, \qquad \kappa^{(i)}_{n+1m} \neq \kappa^{(i)}_{nm}, \qquad \kappa^{(i)}_{nm+1} \neq \kappa^{(i)}_{nm},$$

$$\xi = \mathcal{D}\left[u_{nm}\, \mathrm{e}^{-a_{nm} t}, \kappa^{(1)}_{n+1m}, \kappa^{(2)}_{nm+1}, \kappa^{(3)}_{n-1m+1}, \kappa^{(4)}_{n-1m}, \kappa^{(5)}_{nm-1}, 1_{n+1m-1}\right].$$

Theorem 4.7. *Equation (1) allows a eight-dimensional Abelian symmetry algebra for two classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$A_{8,1}$: $\widehat{X}_i = \lambda^{(i)}_{nm}(t) \partial_{u_{nm}}, \quad i = 1, \ldots, 4; \qquad \widehat{Y}_k = \left(\sum_{j=1}^{4} \omega_{kj}(t) \lambda^{(j)}_{nm}(t) \right) \partial_{u_{nm}}, \quad k = 1, \ldots, 4$

$$F_{nm} = 1 + \begin{vmatrix} u_{nm} & 1 & u_{n+1m} & u_{nm+1} & u_{n-1m+1} \\ \lambda^{(1)}_{nm} & \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{n+1m} & \lambda^{(1)}_{nm+1} & \lambda^{(1)}_{n-1m+1} \\ \lambda^{(2)}_{nm} & \ddot{\lambda}^{(2)}_{nm} & \lambda^{(2)}_{n+1m} & \lambda^{(2)}_{nm+1} & \lambda^{(2)}_{n-1m+1} \\ \lambda^{(3)}_{nm} & \ddot{\lambda}^{(3)}_{nm} & \lambda^{(3)}_{n+1m} & \lambda^{(3)}_{nm+1} & \lambda^{(3)}_{n-1m+1} \\ \lambda^{(4)}_{nm} & \ddot{\lambda}^{(4)}_{nm} & \lambda^{(4)}_{n+1m} & \lambda^{(4)}_{nm+1} & \lambda^{(4)}_{n-1m+1} \end{vmatrix} \left(\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}\right]\right)^{-1}$$
$$+ f_{nm}(t, \xi_{pq}), \qquad (p, q) \in \{(n-1, m), (n, m-1), (n+1, m-1)\},$$

$$\xi_{pq} = \mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, \lambda^{(3)}_{n-1m+1}, \lambda^{(4)}_{pq}\right], \quad \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}\right] \neq 0,$$

$$\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}\right] \neq 0, \qquad \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}\right] \neq 0,$$

$$\text{with} \quad \sum_{j=1}^{4} \left(\ddot{\omega}_{kj} \lambda^{(j)}_{nm} + 2\dot{\omega}_{kj} \dot{\lambda}^{(j)}_{nm}\right) = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0.$$

$A_{8,2}$: $\widehat{X}_1 = \partial_{u_{nm}}, \qquad \widehat{X}_2 = t\partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \lambda^{(i)}_{nm}(t) \partial_{u_{nm}}, \qquad i = 1, 2, 3$

$$\widehat{Y}_k = \left(\sum_{j=1}^{3} \omega_{kj}(t) \lambda^{(j)}_{nm}(t) + \sigma^{(k)}(t) \right) \partial_{u_{nm}}, \qquad k = 1, 2, 3$$

$$F_{nm} = \frac{\ddot{\lambda}^{(1)}_{nm}}{\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}} (u_{n+1m} - u_{nm}) + \frac{\mathcal{D}_2 \cdot \mathcal{D}\left[u, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, 1_{n-1m+1}\right]}{\left(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\right) \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\right]}$$

$$
- \frac{\mathcal{D}_3 \cdot \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(3)}, 1_{nm+1}\big] \cdot \mathcal{D}\big[u, \lambda_{n+1m}^{(1)}, \lambda_{nm+1}^{(2)}, 1_{n-1m+1}\big]}{\big(\lambda_{n+1m}^{(1)} - \lambda_{nm}^{(1)}\big) \cdot \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, 1_{nm+1}\big] \cdot \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, \lambda_{nm+1}^{(3)}, 1_{n-1m+1}\big]}
$$

$$
- \frac{\mathcal{D}_3 \cdot \mathcal{D}\big[u_{nm}, \lambda_{n+1m}^{(1)}, 1_{nm+1}\big]}{\big(\lambda_{n+1m}^{(1)} - \lambda_{nm}^{(1)}\big) \cdot \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, 1_{nm+1}\big]} + f_{nm}(t, \xi_{pq}),
$$

$$
\xi_{pq} = \mathcal{D}\big[u_{nm}, \lambda_{n+1m}^{(1)}, \lambda_{nm+1}^{(2)}, \lambda_{n-1m+1}^{(3)}, 1_{pq}\big],
$$

$$
(p, q) \in \{(n-1, m), (n, m-1), (n+1, m-1)\},
$$

$$
\mathcal{D}_2 = \begin{vmatrix} \ddot{\lambda}_{nm}^{(1)} & \lambda_{nm}^{(1)} & \lambda_{n+1m}^{(1)} \\ \ddot{\lambda}_{nm}^{(2)} & \lambda_{nm}^{(2)} & \lambda_{n+1m}^{(2)} \\ 0 & 1 & 1 \end{vmatrix}, \qquad \mathcal{D}_3 = \begin{vmatrix} \ddot{\lambda}_{nm}^{(1)} & \lambda_{nm}^{(1)} & \lambda_{n+1m}^{(1)} \\ \ddot{\lambda}_{nm}^{(3)} & \lambda_{nm}^{(3)} & \lambda_{n+1m}^{(3)} \\ 0 & 1 & 1 \end{vmatrix},
$$

$$
\lambda_{n+1m}^{(1)} \neq \lambda_{nm}^{(1)}, \qquad \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, 1_{nm+1}\big] \neq 0, \quad \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, \lambda_{nm+1}^{(3)}, 1_{n-1m+1}\big] \neq 0,
$$

$$
\text{with} \quad \sum_{j=1}^{3} \big(\ddot{\omega}_{kj}\lambda_{nm}^{(j)} + 2\dot{\omega}_{kj}\dot{\lambda}_{nm}^{(j)}\big) + \ddot{\sigma}^{(k)} = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0.
$$

Theorem 4.8. *Equation (1) allows a ten-dimensional Abelian symmetry algebra for two classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$$
A_{10,1}: \quad \widehat{X}_i = \lambda_{nm}^{(i)}(t)\partial_{u_{nm}}, \quad i = 1, \dots, 5; \quad \widehat{Y}_k = \left(\sum_{j=1}^{5} \omega_{kj}(t)\lambda_{nm}^{(j)}(t)\right)\partial_{u_{nm}}, \quad k = 1, \dots, 5 \tag{14}
$$

$$
F_{nm} = 1 + \begin{vmatrix} u_{nm} & 1 & u_{n+1m} & u_{nm+1} & u_{n-1m+1} & u_{n-1m} \\ \lambda_{nm}^{(1)} & \ddot{\lambda}_{nm}^{(1)} & \lambda_{n+1m}^{(1)} & \lambda_{nm+1}^{(1)} & \lambda_{n-1m+1}^{(1)} & \lambda_{n-1m}^{(1)} \\ \lambda_{nm}^{(2)} & \ddot{\lambda}_{nm}^{(2)} & \lambda_{n+1m}^{(2)} & \lambda_{nm+1}^{(2)} & \lambda_{n-1m+1}^{(2)} & \lambda_{n-1m}^{(2)} \\ \lambda_{nm}^{(3)} & \ddot{\lambda}_{nm}^{(3)} & \lambda_{n+1m}^{(3)} & \lambda_{nm+1}^{(3)} & \lambda_{n-1m+1}^{(3)} & \lambda_{n-1m}^{(3)} \\ \lambda_{nm}^{(4)} & \ddot{\lambda}_{nm}^{(4)} & \lambda_{n+1m}^{(4)} & \lambda_{nm+1}^{(4)} & \lambda_{n-1m+1}^{(4)} & \lambda_{n-1m}^{(4)} \\ \lambda_{nm}^{(5)} & \ddot{\lambda}_{nm}^{(5)} & \lambda_{n+1m}^{(5)} & \lambda_{nm+1}^{(5)} & \lambda_{n-1m+1}^{(5)} & \lambda_{n-1m}^{(5)} \end{vmatrix}
$$

$$
\times \big(\mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, \lambda_{nm+1}^{(3)}, \lambda_{n-1m+1}^{(4)}, \lambda_{n-1m}^{(5)}\big]\big)^{-1} + f_{nm}(t, \xi_{pq}), \tag{15}
$$

$$
\xi_{pq} = \mathcal{D}\big[u_{nm}, \lambda_{n+1m}^{(1)}, \lambda_{nm+1}^{(2)}, \lambda_{n-1m+1}^{(3)}, \lambda_{n-1m}^{(4)}, \lambda_{pq}^{(5)}\big], \qquad (p, q) \in \{(n, m-1), (n+1, m-1)\},
$$

$$
\mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, \lambda_{nm+1}^{(3)}, \lambda_{n-1m+1}^{(4)}, \lambda_{n-1m}^{(5)}\big] \neq 0, \qquad \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, \lambda_{nm+1}^{(3)}, \lambda_{n-1m+1}^{(4)}\big] \neq 0,
$$

$$
\mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}, \lambda_{nm+1}^{(3)}\big] \neq 0, \qquad \mathcal{D}\big[\lambda_{nm}^{(1)}, \lambda_{n+1m}^{(2)}\big] \neq 0,
$$

$$
\text{with} \quad \sum_{j=1}^{5} \big(\ddot{\omega}_{kj}\lambda_{nm}^{(j)} + 2\dot{\omega}_{kj}\dot{\lambda}_{nm}^{(j)}\big) = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0. \tag{16}
$$

$$
A_{10,2}: \quad \widehat{X}_1 = \partial_{u_{nm}}, \qquad \widehat{X}_2 = t\partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \lambda_{nm}^{(i)}(t)\partial_{u_{nm}}, \qquad i = 1, \dots, 4
$$

$$
\widehat{Y}_k = \left(\sum_{j=1}^{4} \big(\omega_{kj}(t)\lambda_{nm}^{(j)}(t)\big) + \sigma^{(k)}(t)\right)\partial_{u_{nm}}, \qquad k = 1, \dots, 4
$$

$$
\begin{aligned}
F_{nm} &= \frac{\ddot{\lambda}^{(1)}_{nm}}{\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}}(u_{n+1m} - u_{nm}) - \frac{\mathcal{D}_3 \cdot \mathcal{D}\big[u_{nm}, \lambda^{(1)}_{n+1m}, 1_{nm+1}\big]}{\big(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\big) \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1\big]} \\
&\quad + \left(\mathcal{D}_2 - \frac{\mathcal{D}_3 \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(3)}_{n+1m}, 1_{nm+1}\big]}{\mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\big]}\right) \\
&\quad \times \left(\frac{\mathcal{D}\big[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, 1_{n-1m+1}\big]}{\big(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\big) \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\big]}\right) \\
&\quad + \left(\mathcal{D}_4 - \frac{\mathcal{D}_3 \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(4)}_{n+1m}, 1_{nm+1}\big]}{\mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\big]} - \frac{\mathcal{D}_2 \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(4)}_{nm+1}, 1_{n-1m+1}\big]}{\mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\big]}\right) Z \\
&\quad + \left(\frac{\mathcal{D}_3 \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(3)}_{n+1m}, 1_{nm+1}\big] \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(4)}_{nm+1}, 1_{n-1m+1}\big]}{\mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\big] \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\big]}\right) Z + f_{nm}(t, \xi_{pq}),
\end{aligned}
$$

$$
Z = \frac{\mathcal{D}\big[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, \lambda^{(3)}_{n-1m+1}, 1_{n-1m}\big]}{\big(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\big) \cdot \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, 1_{n-1m}\big]},
$$

$$
\mathcal{D}_2 = \begin{vmatrix} \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{nm} & \lambda^{(1)}_{n+1m} \\ \ddot{\lambda}^{(2)}_{nm} & \lambda^{(2)}_{nm} & \lambda^{(2)}_{n+1m} \\ 0 & 1 & 1 \end{vmatrix}, \qquad \mathcal{D}_3 = \begin{vmatrix} \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{nm} & \lambda^{(1)}_{n+1m} \\ \ddot{\lambda}^{(3)}_{nm} & \lambda^{(3)}_{nm} & \lambda^{(3)}_{n+1m} \\ 0 & 1 & 1 \end{vmatrix},
$$

$$
\mathcal{D}_4 = \begin{vmatrix} \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{nm} & \lambda^{(1)}_{n+1m} \\ \ddot{\lambda}^{(4)}_{nm} & \lambda^{(4)}_{nm} & \lambda^{(4)}_{n+1m} \\ 0 & 1 & 1 \end{vmatrix},
$$

$$
\begin{aligned}
&\xi_{pq} = \mathcal{D}\big[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, \lambda^{(3)}_{n-1m+1}, \lambda^{(4)}_{n-1m}, 1_{pq}\big], \\
&\qquad (p, q) \in \{(n, m-1), (n+1, m-1)\}, \\
&\lambda^{(1)}_{n+1m} \neq \lambda^{(1)}_{nm}, \quad \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\big] \neq 0, \quad \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\big] \neq 0, \\
&\mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, 1_{n-1m}\big] \neq 0, \\
&\text{with} \quad \sum_{j=1}^{4} \big(\ddot{\omega}_{kj}\lambda^{(j)}_{nm} + 2\dot{\omega}_{kj}\dot{\lambda}^{(j)}_{nm}\big) + \ddot{\sigma}^{(k)} = 0, \quad \det(\dot{\omega}_{kj}) \neq 0.
\end{aligned}
$$

Theorem 4.9. *Equation (1) allows a 12-dimensional Abelian symmetry algebra for two classes of interactions F_{nm}. The algebras and interaction functions can be represented as follows:*

$$
A_{12,1}\colon \quad \widehat{X}_i = \lambda^{(i)}_{nm}(t)\partial_{u_{nm}}, \quad i = 1, \ldots, 6; \qquad \widehat{Y}_k = \left(\sum_{j=1}^{6} \omega_{kj}(t)\lambda^{(j)}_{nm}(t)\right)\partial_{u_{nm}}, \quad k = 1, \ldots, 6 \tag{17}
$$

$$F_{nm} = 1 + \begin{vmatrix} u_{nm} & 1 & u_{n+1m} & u_{nm+1} & u_{n-1m+1} & u_{n-1m} & u_{nm-1} \\ \lambda^{(1)}_{nm} & \ddot{\lambda}^{(1)}_{nm} & \lambda^{(1)}_{n+1m} & \lambda^{(1)}_{nm+1} & \lambda^{(1)}_{n-1m+1} & \lambda^{(1)}_{n-1m} & \lambda^{(1)}_{nm-1} \\ \lambda^{(2)}_{nm} & \ddot{\lambda}^{(2)}_{nm} & \lambda^{(2)}_{n+1m} & \lambda^{(2)}_{nm+1} & \lambda^{(2)}_{n-1m+1} & \lambda^{(2)}_{n-1m} & \lambda^{(2)}_{nm-1} \\ \lambda^{(3)}_{nm} & \ddot{\lambda}^{(3)}_{nm} & \lambda^{(3)}_{n+1m} & \lambda^{(3)}_{nm+1} & \lambda^{(3)}_{n-1m+1} & \lambda^{(3)}_{n-1m} & \lambda^{(3)}_{nm-1} \\ \lambda^{(4)}_{nm} & \ddot{\lambda}^{(4)}_{nm} & \lambda^{(4)}_{n+1m} & \lambda^{(4)}_{nm+1} & \lambda^{(4)}_{n-1m+1} & \lambda^{(4)}_{n-1m} & \lambda^{(4)}_{nm-1} \\ \lambda^{(5)}_{nm} & \ddot{\lambda}^{(5)}_{nm} & \lambda^{(5)}_{n+1m} & \lambda^{(5)}_{nm+1} & \lambda^{(5)}_{n-1m+1} & \lambda^{(5)}_{n-1m} & \lambda^{(5)}_{nm-1} \\ \lambda^{(6)}_{nm} & \ddot{\lambda}^{(6)}_{nm} & \lambda^{(6)}_{n+1m} & \lambda^{(6)}_{nm+1} & \lambda^{(6)}_{n-1m+1} & \lambda^{(6)}_{n-1m} & \lambda^{(6)}_{nm-1} \end{vmatrix}$$

$$\times \left(\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}, , \lambda^{(6)}_{nm-1}\right]\right)^{-1} + f_{nm}(t, \xi), \qquad (18)$$

$$\xi = \mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, \lambda^{(3)}_{n-1m+1}, \lambda^{(4)}_{n-1m}, \lambda^{(5)}_{nm-1}, \lambda^{(6)}_{n+1m-1}\right],$$
$$\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}, \lambda^{(6)}_{nm-1}\right] \neq 0,$$
$$\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}\right] \neq 0,$$
$$\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}\right] \neq 0, \qquad \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}\right] \neq 0, \qquad \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}\right] \neq 0.$$

$$\textit{with} \quad \sum_{j=1}^{6} \left(\ddot{\omega}_{kj}\lambda^{(j)}_{nm} + 2\dot{\omega}_{kj}\dot{\lambda}^{(j)}_{nm}\right) = 0, \qquad \det(\dot{\omega}_{kj}) \neq 0. \qquad (19)$$

$A_{12,2}$: $\widehat{X}_1 = \partial_{u_{nm}}, \qquad \widehat{X}_2 = t\partial_{u_{nm}}, \qquad \widehat{X}_{i+2} = \lambda^{(i)}_{nm}(t)\partial_{u_{nm}}, \qquad i = 1, \ldots, 5$

$$\widehat{Y}_k = \left(\sum_{j=1}^{5} \omega_{kj}(t)\lambda^{(j)}_{nm}(t) + \sigma_k(t)\right)\partial_{u_{nm}}, \qquad k = 1, \ldots, 5$$

$$\begin{aligned} F_{nm} = {} & \frac{\ddot{\lambda}^{(1)}_{nm}}{\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}}(u_{n+1m} - u_{nm}) - \frac{\mathcal{D}_3 \cdot \mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1m}, 1_{nm+1}\right]}{\left(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\right) \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right]} \\ & + \left(\mathcal{D}_2 - \frac{\mathcal{D}_3 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(3)}_{n+1m}, 1_{nm+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right]}\right) \\ & \times \left(\frac{\mathcal{D}\left[u_{nm}, \lambda^{(1)}_{n+1m}, \lambda^{(2)}_{nm+1}, 1_{n-1m+1}\right]}{\left(\lambda^{(1)}_{n+1m} - \lambda^{(1)}_{nm}\right) \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\right]}\right) \\ & + \left(\mathcal{D}_4 - \frac{\mathcal{D}_3 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(4)}_{n+1m}, 1_{nm+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right]} - \frac{\mathcal{D}_2 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(4)}_{nm+1}, 1_{n-1m+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\right]}\right) Z \\ & + \left(\frac{\mathcal{D}_3 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(3)}_{n+1m}, 1_{nm+1}\right] \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(4)}_{nm+1}, 1_{n-1m+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right] \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\right]}\right) Z \\ & + \left(\mathcal{D}_4 - \frac{\mathcal{D}_3 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(4)}_{n+1m}, 1_{nm+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right]} - \frac{\mathcal{D}_2 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(4)}_{nm+1}, 1_{n-1m+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\right]}\right) Z_3 \\ & + \left(\frac{\mathcal{D}_3 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(3)}_{n+1m}, 1_{nm+1}\right] \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(4)}_{nm+1}, 1_{n-1m+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right] \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, 1_{n-1m+1}\right]}\right) Z_3 \\ & - \left(\mathcal{D}_2 - \frac{\mathcal{D}_3 \cdot \mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(3)}_{n+1m}, 1_{nm+1}\right]}{\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, 1_{nm+1}\right]}\right) Z_2 \end{aligned}$$

$$+\left(\mathcal{D}_5-\frac{\mathcal{D}_1\cdot\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(5)},1_{nm+1}\right]}{\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},1_{nm+1}\right]}\right)Z_1+f_{nm}(t,\xi),$$

$$\xi=\mathcal{D}\left[u_{nm},\lambda_{n+1m}^{(1)},\lambda_{nm+1}^{(2)},\lambda_{n-1m+1}^{(3)},\lambda_{n-1m}^{(4)},\lambda_{nm-1}^{(5)},1_{n+1m-1}\right].$$

$$Z=\frac{\mathcal{D}\left[u_{nm},\lambda_{n+1m}^{(1)},\lambda_{nm+1}^{(2)},\lambda_{n-1m+1}^{(3)},1_{n-1m}\right]}{\left(\lambda_{n+1m}^{(1)}-\lambda_{nm}^{(1)}\right)\cdot\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(3)},\lambda_{n-1m+1}^{(4)},1_{n-1m}\right]},$$

$$Z_1=\frac{\mathcal{D}\left[u_{nm},\lambda_{n+1m}^{(1)},\lambda_{nm+1}^{(2)},\lambda_{n-1m+1}^{(3)},\lambda_{n-1m}^{(4)},1_{nm-1}\right]}{\left(\lambda_{n+1m}^{(1)}-\lambda_{nm}^{(1)}\right)\cdot\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(3)},\lambda_{n-1m+1}^{(4)},\lambda_{n-1m}^{(5)},1_{nm-1}\right]},$$

$$Z_2=\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(5)},1_{n-1m+1}\right]\cdot Z_1,$$

$$Z_3=\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(3)},\lambda_{n-1m+1}^{(5)},1_{n-1m}\right]\cdot Z_1,$$

$$\mathcal{D}_2=\begin{vmatrix}\ddot\lambda_{nm}^{(1)}&\lambda_{nm}^{(1)}&\lambda_{n+1m}^{(1)}\\\ddot\lambda_{nm}^{(2)}&\lambda_{nm}^{(2)}&\lambda_{n+1m}^{(2)}\\0&1&1\end{vmatrix},\qquad\mathcal{D}_3=\begin{vmatrix}\ddot\lambda_{nm}^{(1)}&\lambda_{nm}^{(1)}&\lambda_{n+1m}^{(1)}\\\ddot\lambda_{nm}^{(3)}&\lambda_{nm}^{(3)}&\lambda_{n+1m}^{(3)}\\0&1&1\end{vmatrix},$$

$$\mathcal{D}_4=\begin{vmatrix}\ddot\lambda_{nm}^{(1)}&\lambda_{nm}^{(1)}&\lambda_{n+1m}^{(1)}\\\ddot\lambda_{nm}^{(4)}&\lambda_{nm}^{(4)}&\lambda_{n+1m}^{(4)}\\0&1&1\end{vmatrix},\qquad\mathcal{D}_5=\begin{vmatrix}\ddot\lambda_{nm}^{(1)}&\lambda_{nm}^{(1)}&\lambda_{n+1m}^{(1)}\\\ddot\lambda_{nm}^{(5)}&\lambda_{nm}^{(5)}&\lambda_{n+1m}^{(5)}\\0&1&1\end{vmatrix},$$

$$\lambda_{n+1m}^{(1)}\neq\lambda_{nm}^{(1)},\qquad\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},1_{nm+1}\right]\neq0,$$

$$\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(3)},1_{n-1m+1}\right]\neq0,\quad\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(3)},\lambda_{n-1m+1}^{(4)},1_{n-1m}\right]\neq0,$$

$$\mathcal{D}\left[\lambda_{nm}^{(1)},\lambda_{n+1m}^{(2)},\lambda_{nm+1}^{(3)},\lambda_{n-1m+1}^{(4)},\lambda_{n-1m}^{(5)},1_{nm-1}\right]\neq0,$$

$$\text{with}\quad\sum_{j=1}^{5}\left(\ddot\omega_{kj}\lambda_{nm}^{(j)}+2\dot\omega_{kj}\dot\lambda_{nm}^{(j)}\right)+\ddot\sigma^{(k)}=0,\qquad\det(\dot\omega_{kj})\neq0.$$

5. Nonsolvable symmetry algebras

The vector fields of equation (1) should have the form (6). We can now ask whether it is possible to obtain simple symmetry algebras from these vector fields. We obtain the following theorem.

Theorem 5.1. *Equation (1) allows only one simple Lie algebra,* sl(2, $\mathbb{R}$)*, given by*

$$NS_{3,1}:\quad\widehat{X}_1=\partial_t,\qquad\widehat{X}_2=t\partial_t+\frac{1}{2}u_{nm}\partial_{u_{nm}},\qquad\widehat{X}_3=t^2\partial_t+tu_{nm}\partial_{u_{nm}},$$
$$F_{nm}=\frac{1}{u_{nm}^3}f_{nm}(\xi_{pq}),\qquad\xi_{pq}=\frac{u_{pq}}{u_{nm}},\qquad(p,q)\in\Gamma\backslash\{(n,m)\},\tag{20}$$
$$[\widehat{X}_1,\widehat{X}_2]=\widehat{X}_1,\qquad[\widehat{X}_1,\widehat{X}_3]=2\widehat{X}_2,\qquad[\widehat{X}_2,\widehat{X}_3]=\widehat{X}_3.$$

We can now look for additional symmetries by considering nonsolvable symmetry algebras for equation (1). A nonsolvable Lie algebra must contain a simple subalgebra, i.e. the Lie algebra sl(2, $\mathbb{R}$) of $NS_{3,1}$ in our case. Therefore, we add new vector fields $\widehat{Y}_i$ of the form (6) to $NS_{3,1}$. These vector fields $\{\widehat{Y}_i\}$ forming the radical of the new Lie algebras. The following theorems give the possible nonsolvable symmetry Lie algebras for equation (1). We only list the radical of the nonsolvable Lie algebras since all of these algebras have sl(2, $\mathbb{R}$) of the form (20) as subalgebra.

Theorem 5.2. *Equation (1) allows a four-dimensional nonsolvable symmetry algebra for 1 class of interaction:*

$$
\begin{aligned}
NS_{4,1}:\quad & \widehat{Y}_1 = a_{nm}u_{nm}\partial_{u_{nm}}, \qquad F_{nm} = u_{nm}[(u_{nm})^{a_{n+1m}}(u_{n+1m})^{-a_{nm}}]^{\frac{4}{\mathcal{D}[a_{nm},1_{n+1m}]}} f_{nm}(\xi_{pq}),\\
& \xi_{pq} = (u_{nm})^{-\mathcal{D}[a_{n+1m},1_{pq}]}(u_{n+1m})^{\mathcal{D}[a_{nm},1_{pq}]}(u_{pq})^{-\mathcal{D}[a_{nm},1_{n+1m}]},\\
& (p,q)\in\Gamma\backslash\{(n,m),(n+1,m)\}.
\end{aligned}
$$

Theorem 5.3. *Equation (1) allows a five-dimensional nonsolvable symmetry algebra for two classes of interactions:*

$$
\begin{aligned}
NS_{5,1}:\quad & \widehat{Y}_1 = a^{(1)}_{nm}u_{nm}\partial_{u_{nm}}, \qquad \widehat{Y}_2 = a^{(2)}_{nm}u_{nm}\partial_{u_{nm}}\\
& F_{nm} = u_{nm}\Big[(u_{nm})^{-\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1}]}(u_{n+1m})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1}]}(u_{nm+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m}]}\Big]^{\frac{4}{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},1_{nm+1}]}}\\
& \qquad \times f_{nm}(\xi_{pq}), \qquad (p,q)\in\Gamma\backslash\{(n,m),(n+1,m),(nm+1)\},\\
& \xi_{pq} = (u_{nm})^{\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},1_{pq}]}(u_{n+1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},1_{pq}]}(u_{nm+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},1_{pq}]}\\
& \qquad \times (u_{pq})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},1_{nm+1}]}.\\
NS_{5,2}:\quad & \widehat{Y}_1 = \partial_{u_{nm}}, \qquad \widehat{Y}_2 = t\partial_{u_{nm}}, \qquad F_{nm} = (\mathcal{D}[u_{nm},1_{n+1m}])^{-3} f_{nm}(\xi_{pq})\\
& \xi_{pq} = \frac{\mathcal{D}[u_{nm},1_{pq}]}{\mathcal{D}[u_{nm},1_{n+1m}]}, \qquad (p,q)\in\Gamma\backslash\{(n,m),(n+1,m)\}.
\end{aligned}
$$

Theorem 5.4. *Equation (1) allows a six-dimensional nonsolvable symmetry algebra for 1 class of interaction:*

$$
NS_{6,1}:\quad \widehat{Y}_1 = a^{(1)}_{nm}u_{nm}\partial_{u_{nm}}, \qquad \widehat{Y}_2 = a^{(2)}_{nm}u_{nm}\partial_{u_{nm}}, \qquad \widehat{Y}_3 = a^{(3)}_{nm}u_{nm}\partial_{u_{nm}}
$$

$$
\begin{aligned}
F_{nm} &= u_{nm}\Big[(u_{nm})^{\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1}]}(u_{n+1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1}]}(u_{nm+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1}]}\\
&\qquad \times (u_{n-1m+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1}]}\Big]^{\alpha_{nm}} f_{nm}(\xi_{pq}),\\
\alpha_{nm} &= \frac{4}{\mathcal{D}\big[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},1_{n-1m+1}\big]}, \qquad (p,q)\in\{(n-1,m),(n,m-1),(n+1,m-1)\}.\\
\xi_{pq} &= (u_{nm})^{-\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},1_{pq}]}(u_{n+1m})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},1_{pq}]}\\
&\qquad \times (u_{nm+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1},1_{pq}]}(u_{n-1m+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},1_{pq}]}\\
&\qquad \times (u_{pq})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},1_{n-1m+1}]}.
\end{aligned}
$$

Theorem 5.5. *Equation (1) allows a seven-dimensional nonsolvable symmetry algebra for two classes of interactions:*

$$
NS_{7,1}:\quad Y_i = a^{(i)}_{nm}u_{nm}\partial_{u_{nm}}, \qquad i=1,\ldots,4; \qquad (p,q)\in\{(n,m-1),(n+1,m-1)\},
$$

$$
\begin{aligned}
F_{nm} &= u_{nm}\Big[(u_{nm})^{-\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m}]}(u_{n+1m})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m}]}\\
&\qquad \times (u_{nm+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m}]}(u_{n-1m+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m}]}\\
&\qquad \times (u_{n-1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1}]}\Big]^{\alpha_{nm}} f_{nm}(\xi_{pq}),\\
\xi_{pq} &= (u_{nm})^{\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},1_{pq}]}(u_{n+1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},1_{pq}]}
\end{aligned}
$$

$$\times (u_{nm+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},1_{pq}]}(u_{n-1m+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m},1_{pq}]}$$
$$\times (u_{n-1m})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},1_{pq}]}(u_{pq})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},1_{n-1m}]},$$

$$\alpha_{nm} = \frac{4}{\mathcal{D}\left[a^{(1)}_{nm}, a^{(2)}_{n+1m}, a^{(3)}_{nm+1}, a^{(4)}_{n-1m+1}, 1_{n-1m}\right]}.$$

$NS_{7,2}$: $\widehat{Y}_1 = \partial_{u_{nm}}$, $\widehat{Y}_2 = t\partial_{u_{nm}}$, $\widehat{Y}_3 = \kappa_{nm}\partial_{u_{nm}}$, $\widehat{Y}_4 = \kappa_{nm}t\partial_{u_{nm}}$,

$\dot{\kappa}_{nm} = 0$, $\kappa_{nm} \neq \kappa_{n+1m}$, $\kappa_{nm} \neq \kappa_{nm+1}$,

$$F_{nm} = \left(\mathcal{D}[u_{nm}, \kappa_{n+1m}, 1_{nm+1}]\right)^{-3} f_{nm}(\xi_{pq}), \qquad \xi_{pq} = \frac{\mathcal{D}[u_{nm}, \kappa_{n+1m}, 1_{pq}]}{\mathcal{D}[u_{nm}, \kappa_{n+1m}, 1_{nm+1}]},$$

$\mathcal{D}[u_{nm}, \kappa_{n+1m}, 1_{nm+1}]$, $(p, q) \in \Gamma\backslash\{(n, m), (n + 1, m), (n, m + 1)\}$.

Theorem 5.6. *Equation (1) allows an eight-dimensional nonsolvable symmetry algebra for one class of interaction:*

$NS_{8,1}$: $\widehat{Y}_i = a^{(i)}_{nm}u_{nm}\partial_{u_{nm}}$, $i = 1, \dots, 5$;

$$F_{nm} = u_{nm}\Big[(u_{nm})^{\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1}]}(u_{n+1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1}]}$$
$$\times (u_{nm+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1}]}(u_{n-1m+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1}]}$$
$$\times (u_{n-1m})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{nm-1}]}(u_{nm-1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{n-1m}]}\Big]^{\alpha_{nm}} f_{nm}(\xi),$$

$$\alpha_{nm} = \frac{4}{\mathcal{D}\left[a^{(1)}_{nm}, a^{(2)}_{n+1m}, a^{(3)}_{nm+1}, a^{(4)}_{n-1m+1}, a^{(5)}_{n-1m}, 1_{nm-1}\right]},$$

$$\xi = (u_{nm})^{-\mathcal{D}[a^{(1)}_{n+1m},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1},1_{n+1m-1}]}(u_{n+1m})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{nm+1},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1},1_{n+1m-1}]}$$
$$\times (u_{nm+1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{n-1m+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1},1_{n+1m-1}]}(u_{n-1m+1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m},a^{(5)}_{nm-1},1_{n+1m-1}]}$$
$$\times (u_{n-1m})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{nm-1},1_{n+1m-1}]}(u_{nm-1})^{\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{n-1m},1_{n+1m-1}]}$$
$$\times (u_{n+1m-1})^{-\mathcal{D}[a^{(1)}_{nm},a^{(2)}_{n+1m},a^{(3)}_{nm+1},a^{(4)}_{n-1m+1},a^{(5)}_{n-1m},1_{nm-1}]}.$$

Theorem 5.7. *Equation (1) allows a nine-dimensional nonsolvable symmetry algebra for one class of interaction:*

$NS_{9,1}$: $\widehat{Y}_1 = \partial_{u_{nm}}$, $\widehat{Y}_2 = t\partial_{u_{nm}}$, $\widehat{Y}_3 = \kappa^{(1)}_{nm}\partial_{u_{nm}}$, $\widehat{Y}_4 = \kappa^{(1)}_{nm}t\partial_{u_{nm}}$,

$\widehat{Y}_5 = \kappa^{(2)}_{nm}\partial_{u_{nm}}$, $\widehat{Y}_6 = \kappa^{(2)}_{nm}t\partial_{u_{nm}}$,

$\dot{\kappa}^{(i)}_{nm} = 0$, $\kappa^{(i)}_{nm} \neq \kappa^{(i)}_{n+1m}$, $\kappa^{(i)}_{nm} \neq \kappa^{(i)}_{nm+1}$, $i = 1, 2$;

$$F_{nm} = \left(\mathcal{D}\left[u_{nm}, \kappa^{(1)}_{n+1m}, \kappa^{(2)}_{nm+1}, 1_{n-1m+1}\right]\right)^{-3} f_{nm}(\xi_{pq}),$$

$$\xi_{pq} = \frac{\mathcal{D}\left[u_{nm}, \kappa^{(1)}_{n+1m}, \kappa^{(2)}_{nm+1}, 1_{pq}\right]}{\mathcal{D}[u_{nm}, \kappa^{(1)}_{n+1m}, \kappa^{(2)}_{nm+1}, 1_{n-1m+1}]},$$
$$(p, q) \in \{(n - 1, m), (n, m - 1), (n + 1, m - 1)\}.$$

Theorem 5.8. *Equation (1) allows a 11-dimensional nonsolvable symmetry algebra for one class of interaction:*

$$\begin{aligned} NS_{11,1}:\ & \widehat{Y}_1 = \partial_{u_{nm}}, \quad \widehat{Y}_2 = t\partial_{u_{nm}}, \quad \widehat{Y}_3 = \kappa^{(1)}_{nm}\partial_{u_{nm}}, \quad \widehat{Y}_4 = \kappa^{(1)}_{nm}t\partial_{u_{nm}}, \\ & \widehat{Y}_5 = \kappa^{(2)}_{nm}\partial_{u_{nm}}, \quad \widehat{Y}_6 = \kappa^{(2)}_{nm}t\partial_{u_{nm}}, \quad \widehat{Y}_7 = \kappa^{(3)}_{nm}\partial_{u_{nm}}, \quad \widehat{Y}_8 = \kappa^{(3)}_{nm}t\partial_{u_{nm}}, \end{aligned} \tag{21}$$

$$\dot{\kappa}_{nm}^{(i)} = 0, \quad i = 1, 2, 3; \qquad F_{nm} = \left(\mathcal{D}\left[u_{nm}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, 1_{n-1m}\right]\right)^{-3} f_{nm}(\xi_{pq}), \tag{22}$$

$$\xi_{pq} = \frac{\mathcal{D}\left[u_{nm}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, 1_{pq}\right]}{\mathcal{D}\left[u_{nm}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, 1_{n-1m}\right]}, \qquad (p, q) \in \{(n, m-1), (n+1, m-1)\}.$$

Theorem 5.9. *Equation (1) allows a 13-dimensional nonsolvable symmetry algebra for one class of interaction:*

$$NS_{13,1}: \quad \widehat{Y}_1 = \partial_{u_{nm}}, \quad \widehat{Y}_2 = t\partial_{u_{nm}}, \quad \widehat{Y}_3 = \kappa_{nm}^{(1)}\partial_{u_{nm}}, \quad \widehat{Y}_4 = \kappa_{nm}^{(1)}t\partial_{u_{nm}}, \quad \widehat{Y}_5 = \kappa_{nm}^{(2)}\partial_{u_{nm}},$$
$$\widehat{Y}_6 = \kappa_{nm}^{(2)}t\partial_{u_{nm}}, \quad \widehat{Y}_7 = \kappa_{nm}^{(3)}\partial_{u_{nm}}, \quad \widehat{Y}_8 = \kappa_{nm}^{(3)}t\partial_{u_{nm}}, \quad \widehat{Y}_9 = \kappa_{nm}^{(4)}\partial_{u_{nm}}, \quad \widehat{Y}_{10} = \kappa_{nm}^{(4)}t\partial_{u_{nm}}, \tag{23}$$

$$F_{nm} = \left(\mathcal{D}\left[u_{nm}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, \kappa_{n-1m}^{(4)}, 1_{nm-1}\right]\right)^{-3} f_{nm}(\xi), \tag{24}$$

$$\dot{\kappa}_{nm}^{(i)} = 0, \qquad i = 1, \ldots, 4; \qquad \xi = \frac{\mathcal{D}\left[u_{nm}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, \kappa_{n-1m}^{(4)}, 1_{n+1m-1}\right]}{\mathcal{D}\left[u_{nm}, \kappa_{n+1m}^{(1)}, \kappa_{nm+1}^{(2)}, \kappa_{n-1m+1}^{(3)}, \kappa_{n-1m}^{(4)}, 1_{nm-1}\right]}.$$

6. Conclusions

Group theoretical methods have been used to classify equation (1) according to their symmetry groups. Abelian and nonsolvable symmetry algebras $\mathcal{L}$ have been considered; the class of linear equations (1) has been excluded. The results of the symmetry classification are summed up in table 1.

Classification given in this paper has applications in solid-state physics. The interest in two-dimensional systems has its origin in possible applications to magnetic systems as well as to absorbed layers. The theoretical models treated so far are mainly spin models and range from Ising systems with competing interactions to planar Heisenberg models [20, 21]. Models involving equation of the form (1) appear in [5–8] where analytic and numeric calculations were performed. Lie point symmetries obtained in this work could be considered to obtain analytical solutions of these models by using symmetries to generate new solutions from a known one or by using the 'symmetry reduction method'. Moreover, some interactions found in this work could be considered as models with appropriate symmetries.

A continuation of this study is in progress. We know that the existence of many symmetries is an indication of integrability. Consequently we can ask ourselves which of the equations that are completely specified by their Lie algebras, and therefore that have many symmetries, are integrable. Completely specified equations to be considered for Abelian and nonsolvable Lie algebras are, respectively, the following:

$$A_{6,3}, \quad A_{7,1}, \quad A_{7,2}, \quad A_{12,1} \quad \text{and} \quad A_{12,2},$$

and

$$NS_{8,1}, \quad \text{and} \quad NS_{13,1}.$$

Finally, a further task is to complete the classification, that is to treat the nilpotent and the solvable Lie algebras.

Table 1. Results of the symmetry classification of equation (1).

dim $\mathcal{L}$	Abelian	Nonsolvable	Total
1	3	0	3
2	4	0	4
3	3	1	4
4	5	1	6
5	3	2	5
6	5	1	6
7	2	2	4
8	2	1	3
9	0	1	1
10	2	0	2
11	0	1	1
12	2	0	2
13	0	1	1

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Appendix

This appendix includes the details of the proof for one Abelian algebra and one nonsolvable algebra in the higher dimensional cases.

For the Abelian Lie algebra we consider the proof of $A_{12,1}$ of theorem 4.9. Since the procedure to obtain this classification is to proceed by dimension (for each type of algebras: Abelian or nonsolvable), we suppose that we already have obtained the algebra $A_{10,1}$ of theorem 4.8. We add a new vector field of the form (6), i.e. $\widehat{Z} = \tau(t)\partial_t + \left[\left(\frac{\dot{\tau}}{2} + a_{nm}\right)u_{nm} + \lambda_{nm}(t)\right]\partial_{u_{nm}}$, to the symmetry algebra (14).

Considering the commutation relations $[\widehat{X}_i, \widehat{Z}] = 0$ and $[\widehat{Y}_i, \widehat{Z}] = 0$ for $i = 1, \ldots, 5$ we obtain

$$\tau\dot{\lambda}^{(i)}_{nm} = \lambda^{(i)}_{nm}\left(\frac{\dot{\tau}}{2} + a_{nm}\right) \qquad \text{and} \qquad \tau\left(\sum_{j=1}^{5}\dot{\omega}_{ij}\lambda^{(j)}_{nm} + \omega_{kj}\dot{\lambda}^{(j)}_{nm}\right) = \left(\sum_{j=1}^{5}\omega_{ij}\lambda^{(j)}_{nm}\right)\left(\frac{\dot{\tau}}{2} + a_{nm}\right). \tag{A.1}$$

We separate the proof in two cases:

(A) The case $\tau = 0$. From the first preceding equations we easily find $\widehat{Z} = \lambda^{(6)}_{nm}(t)\partial_{u_{nm}}$, where $\lambda_{nm} := \lambda^{(6)}_{nm}(t)$ for convenience. We want now to solve the remaining determining equation (7):

$$\ddot{\lambda}^{(6)}_{nm} = \sum_{(p,q)\in\Gamma}\lambda^{(6)}_{pq}\partial_{u_{pq}}F_{nm},$$

where F_{nm} is the interaction (15) of $A_{10,1}$. This equation is equivalent to

$\mathcal{D}\left[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}\right]$

$$\times \sum_{(p,q)\in\Gamma'} \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}, \lambda^{(6)}_{pq}\big]\partial_{\xi_{pq}} f(t, \xi_{pq})$$
$$+\mathcal{D}\big[\ddot{\lambda}^{(1)}_{nm}, \lambda^{(2)}_{nm}, \lambda^{(3)}_{n+1m}, \lambda^{(4)}_{nm+1}, \lambda^{(5)}_{n-1m+1}, \lambda^{(6)}_{n-1m}\big] = 0, \qquad \text{(A.2)}$$

where $\Gamma' := \{(n, m-1), (n+1, m-1)\}$. Using then the method of characteristics we find interaction (18) with a symmetry algebra of dimension 11, where the vector field $\widehat{Z} := \widehat{X}_6 = \lambda^{(6)}_{nm}\partial_{u_{nm}}$ is added to the Lie algebra of $A_{10,1}$.

(B) The case $\tau \neq 0$. By the allowed transformations we choose $\tilde{t}$ such that $\tau\tilde{t} = 1$ which implies $\tau = 1$. Moreover, we choose $Q_{nm}(t)$ for which $a_{nm}Q_{nm}(t) + \lambda_{nm}(t) - \dot{Q}_{nm} = 0$ and we obtain $\widehat{Z} = \partial_t + a_{nm}u_{nm}\partial_{u_{nm}}$. The first equation of (A.1) implies that $\lambda^{(i)}_{nm} = \kappa^{(i)}_{nm}\,\mathrm{e}^{a_{nm}t} i = 1, \ldots, 5$ where $\kappa^{(i)}_{nm}$ is an arbitrary function of n, m. Again here, using P_{nm} in the allowed transformations, we can normalize $\kappa^{(1)}_{nm}$ to 1. Therefore, the Lie algebra we consider contains the subalgebra $A_{6,2}$ and the classification has been already done in this lower dimension.

We are now looking if, for the interaction (18), an additional vector field $\widehat{Z} = \tau(t)\partial_t + \big[\big(\frac{\dot{\tau}}{2} + a_{nm}\big)u_{nm} + \lambda_{nm}(t)\big]\partial_{u_{nm}}$ can be added to the symmetry algebras obtained in case A. The calculations are similar to those presented above. The commutation relations considered are then $[\widehat{X}_i, \widehat{Z}] = 0$ and $[\widehat{Y}_j, \widehat{Z}] = 0$ for $i = 1, \ldots, 6$ and $j = 1, \ldots, 5$. Remaining equation (7) implies the following:

$$\mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}, \lambda^{(6)}_{nm-1}\big]$$
$$\times \sum_{(p,q)\in\Gamma'} \mathcal{D}\big[\lambda^{(1)}_{nm}, \lambda^{(2)}_{n+1m}, \lambda^{(3)}_{nm+1}, \lambda^{(4)}_{n-1m+1}, \lambda^{(5)}_{n-1m}, \lambda^{(6)}_{nm-1}, \lambda^{(7)}_{n+1m-1}\big]\partial_\xi f(t, \xi)$$
$$+\mathcal{D}\big[\ddot{\lambda}^{(1)}_{nm}, \lambda^{(2)}_{nm}, \lambda^{(3)}_{n+1m}, \lambda^{(4)}_{nm+1}, \lambda^{(5)}_{n-1m+1}, \lambda^{(6)}_{n-1m}, \lambda^{(7)}_{nm-1}\big] = 0, \qquad \text{(A.3)}$$

where $\lambda_{nm} := \lambda^{(7)}_{nm}(t)$ and F_{nm} is given by interaction (18). Interaction (18) is invariant when $\widehat{Z}$ is added to algebra of A if $\lambda^{(7)}_{nm}(t) = \sum_{j=1}^{6}\omega_{6j}(t)\lambda^{(j)}_{nm}(t)$. Equation (A.2) then becomes equivalent to condition (19) of $A_{12,1}$. Hence, in this case the vector field $\widehat{Z} := \widehat{Y}_6 = \big(\sum_{j=1}^{6}\omega_{6j}\lambda^{(j)}_{nm}\big)\partial_{u_{nm}}$ is added to the vector field of case A with interaction (18).

We now consider the details of the proof for the highest dimensional nonsolvable case, i.e. for theorem 5.9. We suppose that we already know classification for dimension 11, i.e. theorem 5.8 has already been shown. We add a new vector field of the form $\widehat{Y}_9 = \tau(t)\partial_t + \big[\big(\frac{\dot{\tau}}{2} + a_{nm}\big)u_{nm} + \lambda_{nm}(t)\big]\partial_{u_{nm}}$ to the vector fields (21). The Levi theorem [22, 23] tells us that every finite-dimensional Lie algebra $\mathcal{L}$ is a semidirect sum of a semisimple Lie algebra S and a solvable ideal (the radical R):

$$\mathcal{L} = S \rhd R, \qquad [S, S] = S, \qquad [S, R] \subseteq R, \qquad [R, R] \subset R,$$

such that we have

$$[\widehat{X}_i, \widehat{Y}_9] = \sum_{k=1}^{9}\alpha_{ik}\widehat{Y}_k, \quad i = 1, 2, 3 \qquad \text{and} \qquad [\widehat{Y}_j, \widehat{Y}_9] = \sum_{k=1}^{9}\beta_{jk}\widehat{Y}_k, \quad j = 1, \ldots, 8,$$

where α_{ik}, β_{jk} are real constants. The commutation relation $[\widehat{X}_1, \widehat{Y}_9]$ gives us that

$$\tau(t) = \tau_0\,\mathrm{e}^{\alpha_{19}t}, \qquad \alpha_{19}a_{nm} = 0,$$
$$\lambda_{nm}(t) = \begin{cases} \kappa^{(4)}_{nm}\,\mathrm{e}^{\alpha_{19}t} - \dfrac{1}{\alpha_{19}^2}\big(\alpha_{19}A^{(1)}_{nm} + \alpha_{19}B^{(1)}_{nm}t + B^{(1)}_{nm}\big), & \alpha_{19} \neq 0, \\ \dfrac{1}{2}B^{(1)}_{nm}t^2 + A^{(1)}_{nm}t + \kappa^{(4)}_{nm}, & \alpha_{19} = 0, \end{cases}$$

where

$$A_{nm}^{(i)} := \alpha_{i1} + \alpha_{i3}\kappa_{nm}^{(1)} + \alpha_{i5}\kappa_{nm}^{(2)} + \alpha_{i7}\kappa_{nm}^{(3)},$$
$$B_{nm}^{(i)} := \alpha_{i2} + \alpha_{i4}\kappa_{nm}^{(1)} + \alpha_{i6}\kappa_{nm}^{(2)} + \alpha_{i8}\kappa_{nm}^{(3)},$$

for $i = 1, 2, 3$ (functions with $i = 2, 3$ will also appear in what follows) and $\kappa_{nm}^{(4)}$ is an arbitrary function of n, m. Considering now the commutation relation $[\widehat{X}_2, \widehat{Y}_9]$ we obtain

$$\alpha_{19}\tau_0 = 0, \qquad (\alpha_{29} + 1)\tau_0 = 0, \qquad \alpha_{29}a_{nm} = 0,$$

and

$$\begin{cases} \kappa_{nm}^{(4)} = 0 \\ (2\alpha_{29} - 1)B_{nm}^{(1)} - 2\alpha_{19}B_{nm}^{(2)} = 0 \\ (1 + 2\alpha_{29})\alpha_{19}A_{nm}^{(1)} + (1 + 2\alpha_{29})B_{nm}^{(1)} + 2\alpha_{19}^2 A_{nm}^{(2)} = 0 \end{cases} \quad \text{for} \quad \alpha_{19} \neq 0,$$

$$\begin{cases} \left(\frac{3}{2} - \alpha_{29}\right)\left(B_{nm}^{(1)}\right)^2 = 0 \\ \left(\alpha_{29+\frac{1}{2}}\right)\kappa_{nm}^{(4)} + A_{nm}^{(2)} = 0 \\ \left(\alpha_{29} - \frac{1}{2}\right) A_{nm}^{(1)} + B_{nm}^{(2)} = 0. \end{cases} \quad \text{for} \quad \alpha_{19} = 0.$$

From the commutation relation $[\widehat{X}_3, \widehat{Y}_9]$ we find

$$\tau_0 = 0, \qquad \alpha_{39}a_{nm} = 0$$

and

$$\begin{cases} \alpha_{19}\alpha_{39}A_{nm}^{(1)} - \alpha_{19}^2 A_{nm}^{(3)} + \alpha_{39}B_{nm}^{(1)} = 0 \\ (1 + \alpha_{19}\alpha_{39})B_{nm}^{(1)} - \alpha_{19}^2 B_{nm}^{(3)} + \alpha_{19}A_{nm}^{(1)} = 0 \end{cases} \quad \text{for} \quad \alpha_{19} \neq 0,$$
$$\begin{cases} B_{nm}^{(1)} = 0 \\ \kappa_{nm}^{(4)} + B_{nm}^{(3)} + \alpha_{39}A_{nm}^{(1)} = 0 \\ A_{nm}^{(3)} + \alpha_{39}\kappa_{nm}^{(4)} = 0 \end{cases} \quad \text{for} \quad \alpha_{19} = 0.$$

Finally, for $j = 1, \ldots, 8$ the commutation relations $[\widehat{Y}_j, \widehat{Y}_9]$ imply $\beta_{j9}a_{nm} = 0$ and

$$\begin{cases} \kappa^{(\frac{j-1}{2})}a_{nm} = C_{nm}^{(j)} + \beta_{j9}\kappa_{nm}^{(4)} \\ D_{nm}^{(j)} + \beta_{j9}A_{nm}^{(1)} = 0 \end{cases} \quad \text{for} \quad j = 1, 3, 5, 7$$
$$\begin{cases} \kappa^{(\frac{j-2}{2})}a_{nm} = D_{nm}^{(j)} + \beta_{j9}A_{nm}^{(1)} \\ C_{nm}^{(j)} + \beta_{j9}\kappa_{nm}^{(4)} = 0 \end{cases} \quad \text{for} \quad j = 2, 4, 6, 8,$$

where

$$C_{nm}^{(j)} := \beta_{j1} + \beta_{j3}\kappa_{nm}^{(1)} + \beta_{j5}\kappa_{nm}^{(2)} + \beta_{j7}\kappa_{nm}^{(3)},$$
$$D_{nm}^{(j)} := \beta_{j2} + \beta_{j4}\kappa_{nm}^{(1)} + \beta_{j6}\kappa_{nm}^{(2)} + \beta_{j8}\kappa_{nm}^{(3)},$$

and $\kappa_{nm}^{(0)} := 1$ for convenience.

Since $A_{nm}^{(i)}, B_{nm}^{(i)}, C_{nm}^{(j)}$ and $D_{nm}^{(j)}$ are linear combinations of linearly independent functions appearing in the vector fields $\widehat{Y}_j$, we can use linear combinations to simplify $\widehat{Y}_9$. In the generic case, i.e. for any α_{19}, we have that $a_{nm} = C_{nm}^{(1)}$ and using linear combinations we can transform a_{nm} to zero. Moreover, we have that $\tau = 0$ such that $\widehat{Y}_9 = \lambda_{nm}(t)\partial_{u_{nm}}$, where λ_{nm} depends on α_{19}. In the case of $\alpha_{19} \neq 0$ we can transform $A_{nm}^{(1)} = B_{nm}^{(1)} = 0$ by linear combinations. From equations obtained previously, this implies that $A_{nm}^{(i)} = B_{nm}^{(i)} = 0$ for $i = 1, 2, 3$. Since $\kappa_{nm}^{(4)} = 0$, no additional symmetry is possible when $\alpha_{19} \neq 0$. When $\alpha_{19} = 0$, one equation gives us that $B_{nm}^{(1)} = 0$ and by linear combinations we can transform $A_{nm}^{(1)}$ to zero. Therefore, we find

$$\widehat{Y}_9 = \kappa_{nm}^{(4)}\partial_{u_{nm}}.$$

Note that we cannot use allowed transformations to simplify $\widehat{Y}_9$. All allowed transformations have already been used to simplify vector fields, in particular to obtain $\kappa_{nm}^{(0)} = 1$ in $\widehat{Y}_1$ and $\widehat{Y}_2$.

From the remaining equation (7) we have

$$0 = \sum_{(p,q)\in\Gamma} \kappa_{pq}^{(4)} \partial_{u_{pq}} F_{nm},$$

where F_{nm} is given by (22). Using the method of characteristics we find that the new invariant function when $\widehat{Y}_9$ is added to $NS_{11,1}$ is given by (24).

Let us now verify that the Lie algebra $\{\widehat{X}_1, \widehat{X}_2, \widehat{X}_3, \widehat{Y}_1, \ldots, \widehat{Y}_9\}$ is 'maximal' or not, i.e. if we can add another vector field of the form $\widehat{Y}_{10} = \tau(t)\partial_t + \left[\left(\frac{\dot{\tau}}{2} + a_{nm}\right) u_{nm} + \lambda_{nm}(t)\right] \partial_{u_{nm}}$ with the same invariant function (24). From the Levi theorem, we have

$$[\widehat{X}_i, \widehat{Y}_{10}] = \sum_{k=1}^{10} \widetilde{\alpha}_{ik} \widehat{Y}_k, \qquad i = 1, 2, 3 \qquad \text{and} \qquad [\widehat{Y}_j, \widehat{Y}_{10}] = \sum_{k=1}^{10} \widetilde{\beta}_{jk} \widehat{Y}_k, \quad j = 1, \ldots, 9,$$

where $\widetilde{\alpha}_{ik}, \widetilde{\beta}_{jk}$ are real constants. Now defining

$$\begin{aligned}
\widetilde{A}_{nm}^{(i)} &:= \widetilde{\alpha}_{i1} + \widetilde{\alpha}_{i3}\kappa_{nm}^{(1)} + \widetilde{\alpha}_{i5}\kappa_{nm}^{(2)} + \widetilde{\alpha}_{i7}\kappa_{nm}^{(3)} + \widetilde{\alpha}_{i9}\kappa_{nm}^{(4)},\\
\widetilde{B}_{nm}^{(i)} &:= \widetilde{\alpha}_{i2} + \widetilde{\alpha}_{i4}\kappa_{nm}^{(1)} + \widetilde{\alpha}_{i6}\kappa_{nm}^{(2)} + \widetilde{\alpha}_{i8}\kappa_{nm}^{(3)},\\
\widetilde{C}_{nm}^{(j)} &:= \widetilde{\beta}_{j1} + \widetilde{\beta}_{j3}\kappa_{nm}^{(1)} + \widetilde{\beta}_{j5}\kappa_{nm}^{(2)} + \widetilde{\beta}_{j7}\kappa_{nm}^{(3)} + \widetilde{\beta}_{j9}\kappa_{nm}^{(4)},\\
\widetilde{D}_{nm}^{(j)} &:= \widetilde{\beta}_{j2} + \widetilde{\beta}_{j4}\kappa_{nm}^{(1)} + \widetilde{\beta}_{j6}\kappa_{nm}^{(2)} + \widetilde{\beta}_{j8}\kappa_{nm}^{(3)},
\end{aligned}$$

the commutation relations $[\widehat{X}_i, \widehat{Y}_{10}]$ and $[\widehat{Y}_j, \widehat{Y}_{10}]$ give us the same set of equations obtained previously replacing $A_{nm}^{(i)} \to \widetilde{A}_{nm}^{(i)}, \ldots, D_{nm}^{(j)} \to \widetilde{D}_{nm}^{(j)}, \alpha_{ik} \to \widetilde{\alpha}_{ik}, \beta_{jk} \to \widetilde{\beta}_{jk}$ and $\kappa_{nm}^{(4)} \to \widetilde{\kappa}_{nm}^{(4)}$ for $i = 1, 2, 3$, $j = 1, \ldots, 9$ and $k = 1, \ldots, 10$. Again here we have that $\tau = 0$ and $a_{nm} = \widetilde{C}_{nm}^{(1)}$. Using linear combinations we can transform a_{nm} to zero such that $\widehat{Y}_{10} = \lambda_{nm}(t)\partial_{u_{nm}}$. For $\alpha_{110} = 0$ we can transform λ_{nm} to zero by linear combination. In the case $\alpha_{110} \neq 0$ we can transform $\widetilde{A}_{nm}^{(1)}$ to zero by linear combinations but not $\widetilde{B}_{nm}^{(1)}$ (since this function does not depend on $\kappa_{nm}^{(4)}$). Therefore, we obtain

$$\widehat{Y}_{10} = \kappa_{nm}^{(4)} t \partial_{u_{nm}}.$$

The remaining equation (7)

$$0 = \sum_{(p,q)\in\Gamma} \kappa_{pq}^{(4)} t \partial_{u_{pq}} F_{nm},$$

with F_{nm} given by (24), is identically zero. This completes the proof for the nonsolvable case of theorem 5.9.

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